§ 10.4 Continued.

EXAMPLE

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardoid $r = 1 + \sin\theta$.

SOLUTION

- First sketch the graph.

$r = 3\sin\theta$

$r^2 = 3r\sin\theta$

$\cancel{r^2 = 3r}$ $\quad x^2 + y^2 = 3y$

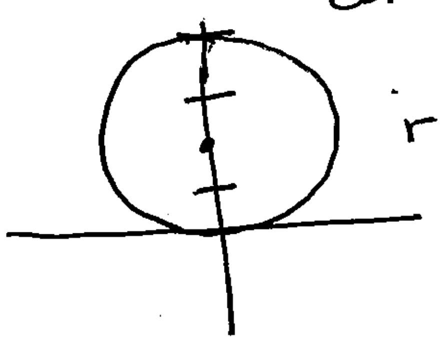
$$x^2 + y^2 - 3y = 0$$

$$x^2 + \left(y^2 - 3y + \tfrac{9}{4}\right) = \tfrac{9}{4}$$

$$\left(-\tfrac{3}{2}\right)^2$$

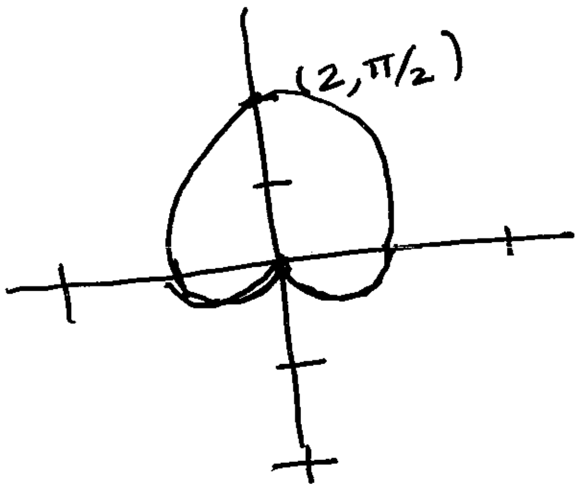
$$x^2 + \left(y - \tfrac{3}{2}\right)^2 = \left(\tfrac{3}{2}\right)^2$$

center $(0, 3/2)$, $r = 3/2$.

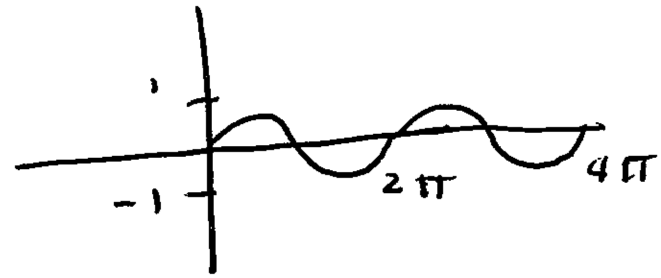


$r = 3\sin\theta$

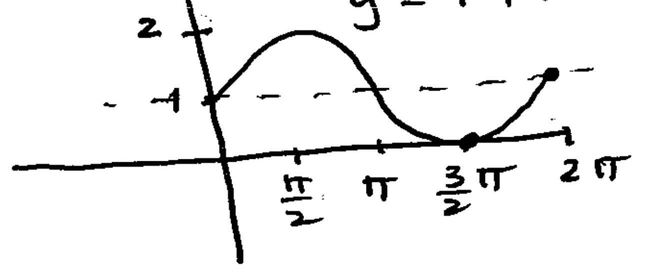
$r = 1 + \sin\theta$



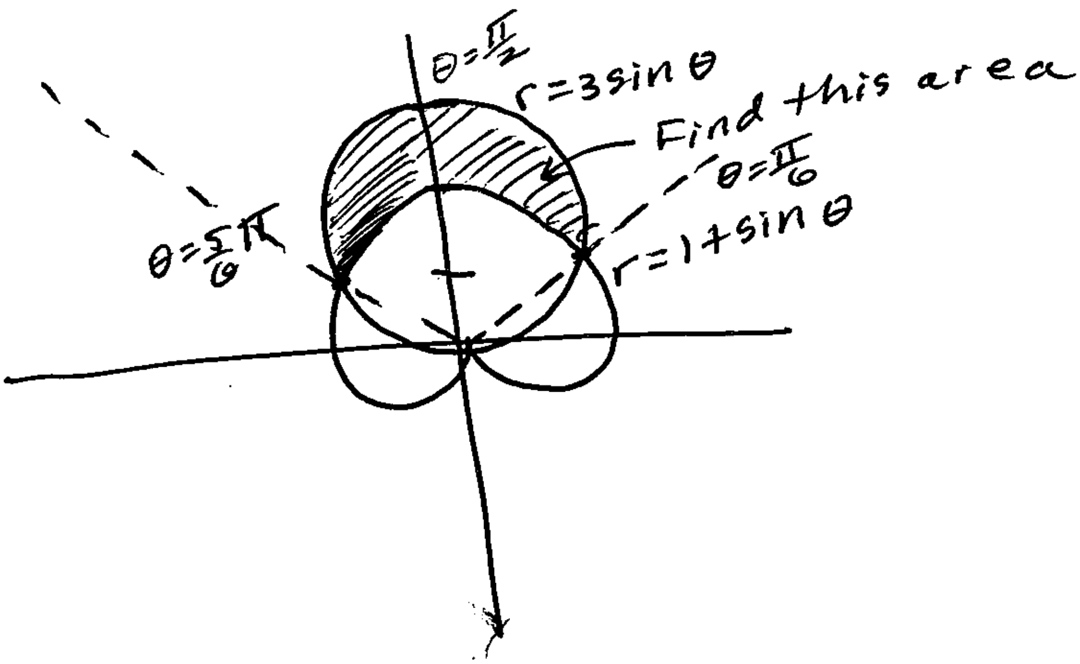
$y = \sin x$



$y = 1 + \sin x$



θ	r
0	1
$\frac{\pi}{2}$	2
π	1
$\frac{3}{2}\pi$	0
2π	1



Points of intersection

$r = 3\sin\theta$

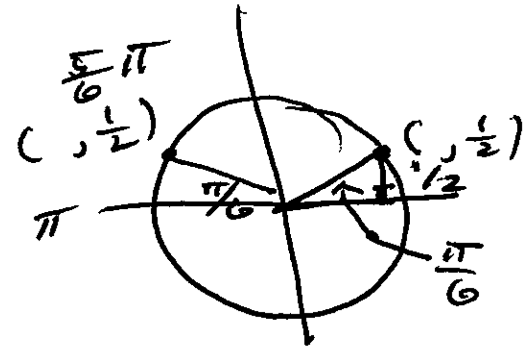
$r = 1 + \sin\theta$

$3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ or $\theta = \frac{5}{6}\pi$



Area of a region

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_\alpha^b \frac{1}{2} r^2 d\theta$$

Area = Area of circle − Area Cardoid.
(between $\pi/6$ and $5/6\,\pi$)

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(3\sin\theta)^2 d\theta - \int_{\frac{\pi}{6}}^{5/6\,\pi} \frac{1}{2}(1+\sin\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{6}}^{5/6\,\pi} \frac{1}{2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

$$= 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/2} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\pi/2} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta$$

$\boxed{\text{Recall } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)}$

$$= \int_{\pi/6}^{\pi/2} \left[8\cdot\frac{1}{2}(1-\cos 2\theta) - 2\sin\theta - 1\right] d\theta$$

$$= \left[4\left(\theta - \frac{\sin 2\theta}{2}\right) - 2(-\cos\theta) - \theta\right]_{\pi/6}^{\pi/2}$$

$$= 3\left[\theta\right]_{\pi/6}^{\pi/2} - 2\left[\sin 2\theta\right]_{\pi/6}^{\pi/2} + 2\left[\cos\theta\right]_{\pi/6}^{\pi/2}$$

$\sin\pi = 0$

$$= 3\left[\pi/2 - \frac{\pi}{6}\right] - 2\left[\sin\left(2\cdot\frac{\pi}{2}\right) - \sin\left(2\cdot\frac{\pi}{6}\right)\right]$$

$$+ 2\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{6}\right]$$

$$= \cancel{3\left[\frac{\pi}{3}\right] - 2\left[\theta = \right.}$$

$$= 3\left[\frac{3}{6}\pi - \frac{\pi}{6}\right] - 2\left[\sin\pi - \sin\frac{\pi}{3}\right]$$

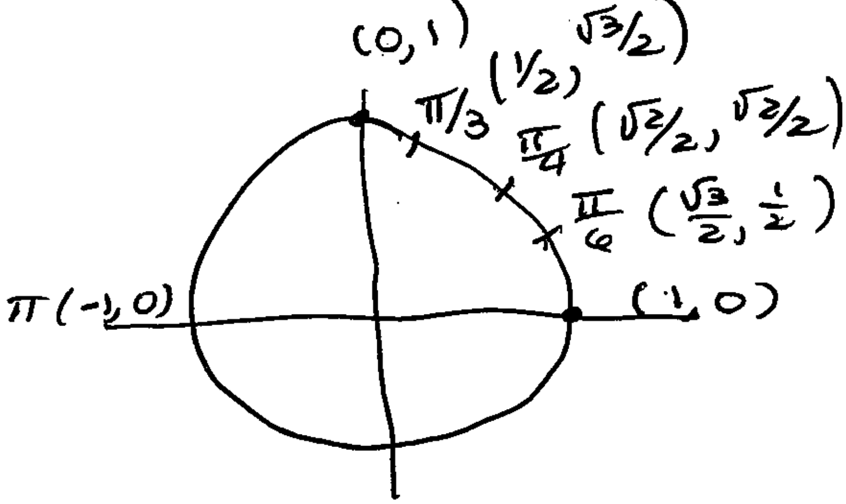
$$+ 2\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{6}\right]$$

$$= 3\left[\frac{\pi}{3}\right] - 2\left[0 - \frac{\sqrt{3}}{2}\right]$$

$$+ 2\left[0 - \frac{\sqrt{3}}{2}\right]$$

$$= \pi + \sqrt{3} - \sqrt{3}$$

$$= \pi$$



Arc Length

we have $r = f(\theta)$
$x = r\cos\theta$
$y = r\sin\theta$
so
$x = f(\theta)\cos\theta$
$y = f(\theta)\sin\theta$

$$L = \int ds$$

$$ds^2 = dx^2 + dy^2$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$L = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Find $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$

$x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cancel{\sin\theta}\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \cancel{dy} \left(\frac{dy}{d\theta}\right)^2$$

$$= \left[f'(\theta)\cos\theta - f(\theta)\sin\theta\right]^2 + \left[f'(\theta)\sin\theta + f(\theta)\cos\theta\right]^2$$

$$= (f'(\theta)\cos\theta)^2 - \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta} + (f(\theta))^2\sin^2\theta + (f'(\theta))^2\sin^2\theta + \cancel{2f'(\theta)f(\theta)\sin\theta\cos\theta} + (f(\theta))^2\cos^2\theta$$

$$= (f'(\theta))^2 (\underbrace{\cos^2\theta + \sin^2\theta}_{=1})$$

$$+ (f(\theta))^2 (\underbrace{\cos^2\theta + \sin^2\theta}_{=1})$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

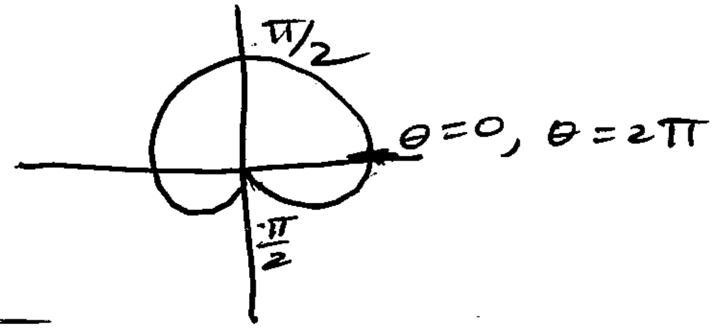
$$= (f'(\theta))^2 + (f(\theta))^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$L = \int ds$$

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

EXAMPLE Find the length of the cardoid $r = 1 + \sin\theta$.



$$L = \int_a^b \sqrt{r^2 + (r')^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

Aside

$r = 1 + \sin\theta$

$\frac{dr}{d\theta} = \cos\theta$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + 1}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(2 + 2\sin\theta)\left(\frac{2 - 2\sin\theta}{2 - 2\sin\theta}\right)}\, d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2 - 2\sin\theta}}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\pi/2} \frac{2\sqrt{1-\sin^2\theta}}{\sqrt{2-2\sin\theta}}\, d\theta$$

$$\sqrt{x^2} = |x|$$

$$= 2\int_{-\pi/2}^{\pi/2} \frac{2\sqrt{\cos^2\theta}}{\sqrt{2-2\sin\theta}}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\, d\theta$$

$u = 2-2\sin\theta$

$du = -2\cos\theta\, d\theta$

$-\frac{1}{2}\, du = \cos\theta\, d\theta$

$$= 2\int_{-\frac{\pi}{2}}^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta$$

$\theta = \frac{-\pi}{2}$

$u = 2-2\sin(-\frac{\pi}{2})$

$u = 2-2(-1)$

$= 4$

$\theta = \pi/2$

$u = 2-2\sin(\frac{\pi}{2})$

$= 2-2$

$= 0$

$$= 2\left(-\tfrac{1}{2}\right)2\int_4^0 \frac{1}{\sqrt{u}}\, du$$

$$= -2\int_4^0 u^{-1/2}\, du$$

$$= 2\int_0^4 u^{-1/2}\, du$$

$$= 2\left.\frac{u^{1/2}}{1/2}\right|_0^4 = 4\left[\sqrt{u}\right]_0^4$$

$$= 4\left(\sqrt{4}-\sqrt{0}\right)$$

$$= 4(2) = 8$$

Homework Review

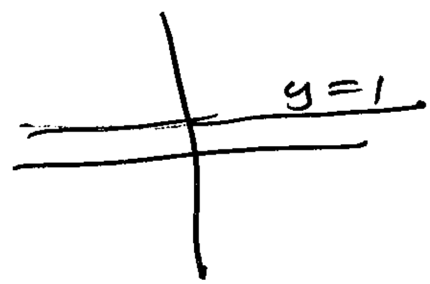
§10.3 Identify the curve by finding a Cartesian equation for the curve.

#19 $r = \csc\theta$

$r = \dfrac{1}{\sin\theta}$

$r\sin\theta = 1$

$y = 1$



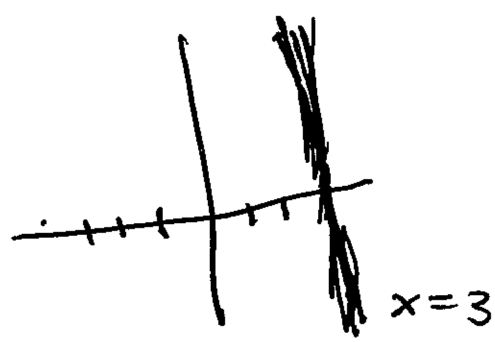
Follow up

$r = 3\sec\theta$

$r = \dfrac{3}{\cos\theta}$

$r\cos\theta = 3$

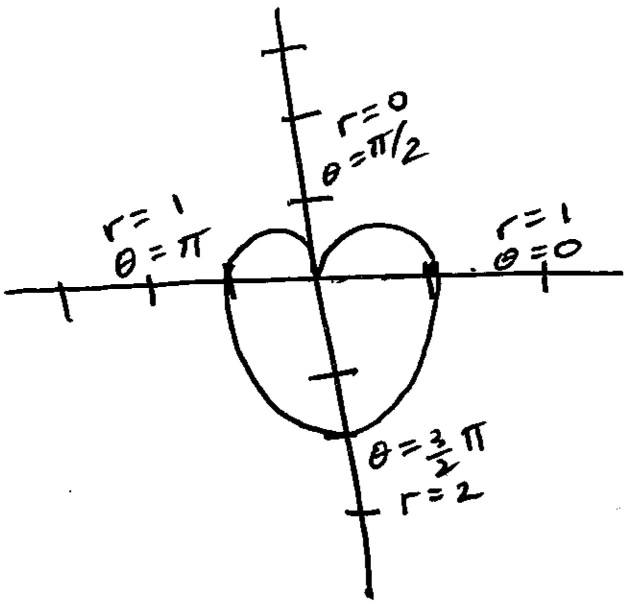
$x = 3$



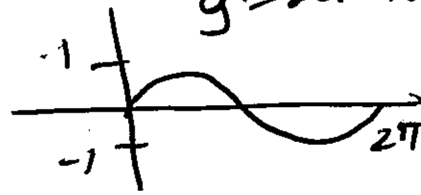
§ 10.3 #33

$$r = 2(1 - \sin\theta), \quad \theta \geq 0$$

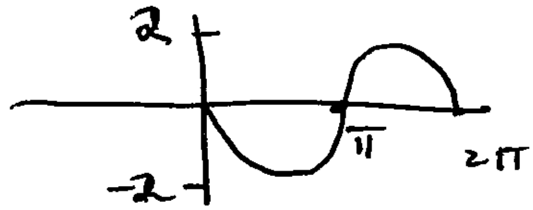
$$r = 2 - 2\sin\theta$$



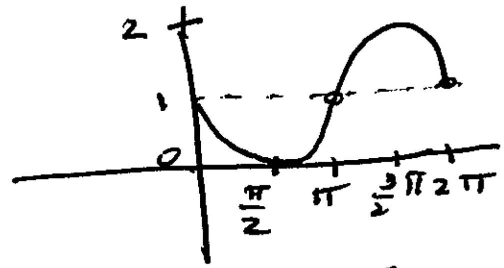
$y = \sin x$



$y = -2\sin x$



$y = 2 - 2\sin x$



θ	r
0	1
$\frac{\pi}{2}$	0
π	1
$\frac{3}{2}\pi$	2
2π	1