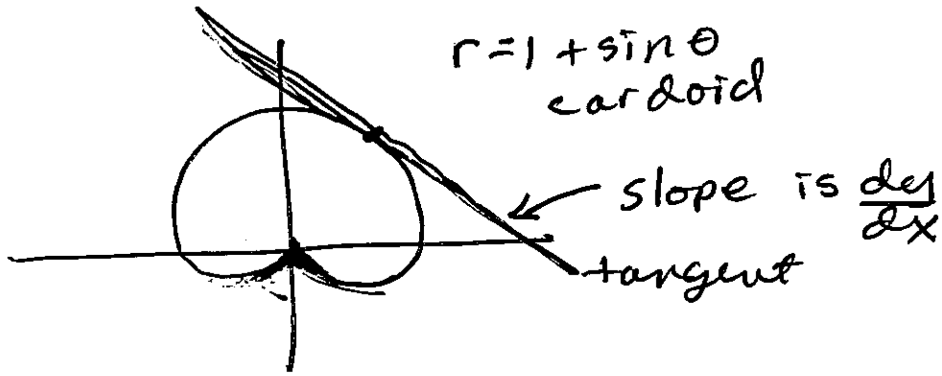
§10.3 continued

Additional HW §10.3 # 57–67 odd

Tangents to Polar Curves.



$x = r\cos\theta$
$y = r\sin\theta$

Assume that $r = f(\theta)$ is a function of θ.

so $x = f(\theta)\cos\theta$
$y = f(\theta)\sin\theta$

We have from §10.1

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(f(\theta)\sin\theta)'}{(f(\theta)\cos\theta)'}$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

or write as

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

EXAMPLE a) For the cardoid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

SOLUTION

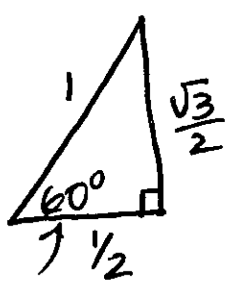
$$x = r\cos\theta = (1+\sin\theta)\cos\theta$$

$$y = r\sin\theta = (1+\sin\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{[(1+\sin\theta)\sin\theta]'}{[(1+\sin\theta)\cos\theta]'}$$

$$= \frac{(\cos\theta)(\sin\theta) + (1+\sin\theta)\cos\theta}{(\cos\theta)(\cos\theta) + (1+\sin\theta)(-\sin\theta)}$$

$\theta = \pi/3$



$\frac{\pi}{3}$ $\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

at $\theta = \pi/3$

$$= \frac{(1/2)(\sqrt{3}/2) + (1 + \sqrt{3}/2)(\frac{1}{2})}{(1/2)(1/2) + (1+\sqrt{3}/2)(-\frac{\sqrt{3}}{2})}$$

$$= \left(\frac{\sqrt{3}/4 + 1/2 + \sqrt{3}/4}{1/4 - \sqrt{3}/2 - 3/4}\right)\frac{4}{4}$$

$$= \frac{\sqrt{3} + 2 + \sqrt{3}}{1 - 2\sqrt{3} - 3}$$

$$= \frac{2\sqrt{3} + 2}{-2\sqrt{3} - 2} = \frac{\cancel{2}(\sqrt{3}+1)}{-\cancel{2}(+\sqrt{3}+1)}$$

$$= \frac{(\sqrt{3}+1)}{-(\sqrt{3}+1)} = -1$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = -1$$

(b) Find the points on the cardoid where the tangent line is horizontal or vertical.

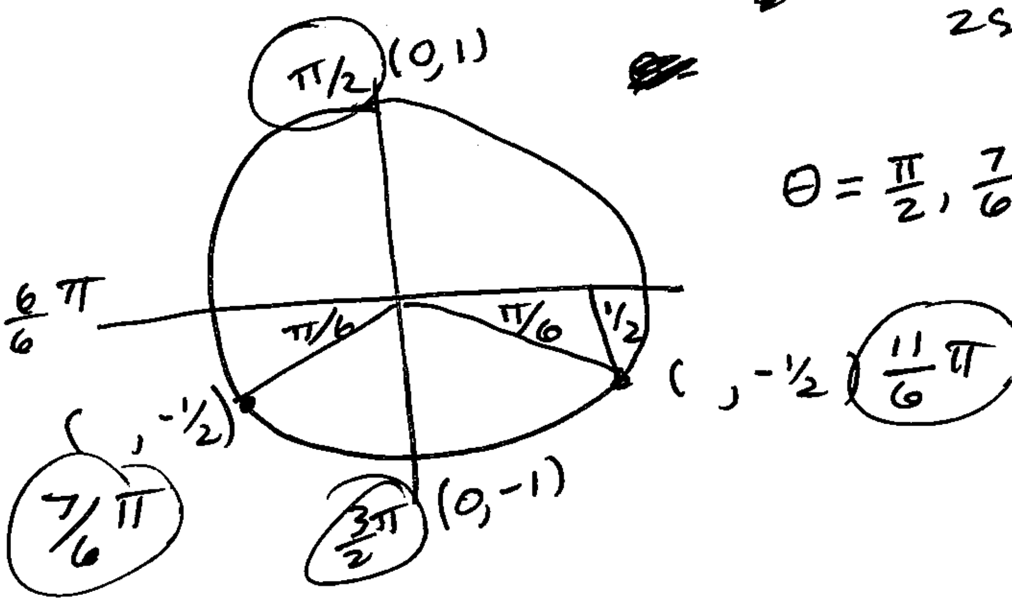
SOLUTION horizontal tangent line

Solve $\frac{dy}{dx} = 0$

i.e. $\frac{dy/d\theta}{dx/d\theta} = 0$ Find where $\frac{dy}{d\theta} = 0$.

$$\frac{dy}{d\theta} = \cos\theta \sin\theta + (1+\sin\theta)\cos\theta$$
$$= \cos\theta \left(\sin\theta + 1 + \sin\theta \right)$$
$$= \cos\theta \, (2\sin\theta + 1)$$

$$\cos\theta\,(2\sin\theta+1) = 0$$
$$\cos\theta = 0 \text{ or } 2\sin\theta + 1 = 0$$
$$2\sin\theta = -1$$
$$\sin\theta = -\tfrac{1}{2}$$



$$\theta = \frac{\pi}{2}, \frac{7}{6}\pi, \frac{3}{2}\pi, \frac{11}{6}\pi$$

θ	$r = 1+\sin\theta$
$\pi/2$	$1+\sin\pi/2 = 1+1=2$
$\frac{7}{6}\pi$	$1+\sin\frac{7}{6}\pi = 1+-\frac{1}{2} = \frac{1}{2}$
$\frac{3}{2}\pi$	$1+\sin\frac{3}{2}\pi = 1+-1 = 0$
$\frac{11}{6}\pi$	$1+\sin\frac{11}{6}\pi = 1+-\frac{1}{2} = \frac{1}{2}$

<u>points</u> (in polar coord) $(2, \pi/2), (1/2, \frac{7}{6}\pi), (0, 3/2\,\pi), (1/2, \frac{11}{6}\pi)$

The tangent line is vertical.

<u>solve</u> $\frac{dx}{dy} = 0$

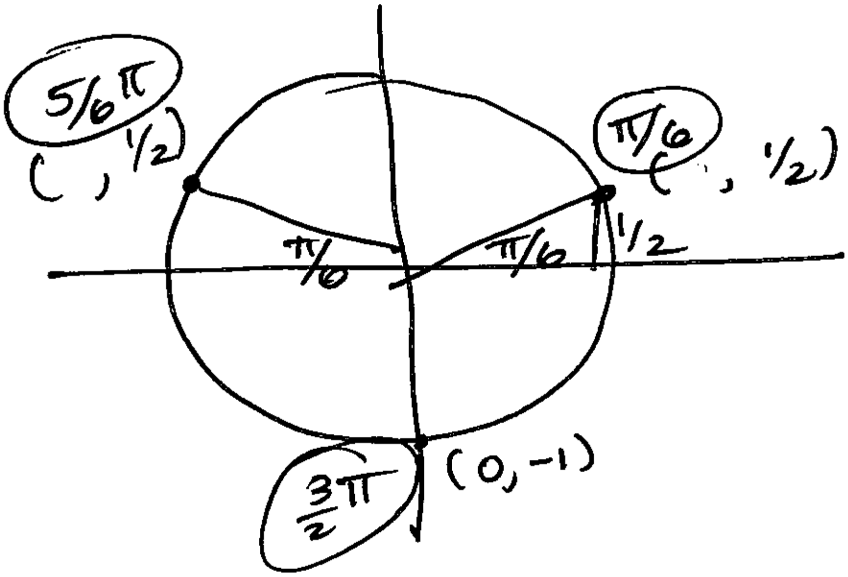
i.e. $\frac{dx/d\theta}{dy/d\theta} = 0$

i.e. $dx/d\theta = 0$

$$\cos^2\theta + (1+\sin\theta)(-\sin\theta) = 0$$
$$\cos^2\theta - \sin\theta - \sin^2\theta = 0$$
$$1-\sin^2\theta - \sin\theta - \sin^2\theta = 0$$
$$-2\sin^2\theta - \sin\theta + 1 = 0$$

$u = \sin\theta$

$$-2u^2 - u + 1 = 0$$
$$2u^2 + u - 1 = 0$$
$$(2u-1)(u+1) = 0$$
$$2u-1=0, \quad u+1=0$$
$$2u=1 \quad u=-1$$
$$u = 1/2$$
$$\sin\theta = 1/2, \quad \sin\theta = -1$$



θ	$r = 1 + \sin\theta$
$\frac{\pi}{6}$	$1 + \sin\frac{\pi}{6} = 1 + 1/2$ $= 3/2$
$\frac{5}{6}\pi$	$1 + \sin\frac{5}{6}\pi = 1 + \frac{1}{2}$ $= \frac{3}{2}$
$\frac{3}{2}\pi$	$1 + \sin\frac{3}{2}\pi = 1 - 1 = 0$

~~points ($3/2$, π~~ points $(3/2, \frac{\pi}{6}), (3/2, \frac{5}{6}\pi),$
$(0, 3/2\pi)$.

<u>Note</u>:

At $\theta = \frac{3}{2}\pi$

$$\frac{dy}{d\theta} = 0 \quad \text{and} \quad \frac{dx}{d\theta} = 0$$

so $\frac{dy}{dx} = \frac{0}{0}$ undef.

So to find what is happening here, we find the limit

$$\lim_{\theta \to (\frac{3}{2}\pi)^-} \frac{dy}{dx}$$

$$= \lim_{\theta \to (\frac{3}{2}\pi)^-} \frac{\cos\theta \sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta + (1+\sin\theta)(-\sin\theta)}$$

$$= \lim_{\theta \to (\frac{3}{2}\pi)^-} \frac{\cos\theta\sin\theta + \cos\theta + \cos\theta\sin\theta}{\cos^2\theta - \sin\theta - \sin^2\theta}$$

$$= \lim_{\theta \to \frac{3}{2}\pi^-} \frac{2\cos\theta\sin\theta + \cos\theta}{\cos^2\theta - \sin\theta - \sin^2\theta} \qquad \frac{0}{0}$$

$$\cancel{= \lim_{\theta \to \frac{3}{2}\pi} \frac{\cos\theta(2\sin\theta + 1)}{1 - \sin^2\theta - \sin\theta - \sin^2\theta}}$$

$$\cancel{= \lim_{\theta \to \frac{3}{2}\pi} \frac{\cos\theta(2\sin\theta + 1)}{\sin^2\theta - \sin\theta + 1}}$$

$$\cancel{= \lim \frac{\cos\theta(2\sin\theta + 1)}{-(2\sin\theta + 1)(\sin\theta - 1)}}$$

$$\overset{LH}{=} \lim_{\theta \to \frac{3}{2}\pi^-} \frac{2(-\sin\overset{\nearrow -1}{\theta})\sin\overset{\nearrow -1}{\theta} + 2\cos\theta\cos\overset{\nearrow 0}{\theta} - \sin\overset{\nearrow -1}{\theta}}{2\underset{\searrow 0}{\cos\theta}(-\sin\theta) - \underset{\searrow 0}{\cos\theta} - 2\underset{\searrow -1}{\sin\theta}\underset{\searrow 0}{\cos\theta}}$$

$$= \frac{2+1}{0} \text{ form}$$

$$= \infty$$

Vertical tangent line at $\theta = \frac{3}{2}\pi$

Answer: horizontal:

$\left(2, \frac{\pi}{2}\right), \left(\frac{1}{2}, \frac{7}{6}\pi\right), \left(\frac{1}{2}, \frac{11}{6}\pi\right)$

Vertical

$\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5}{6}\pi\right), \left(0, \frac{3}{2}\pi\right)$

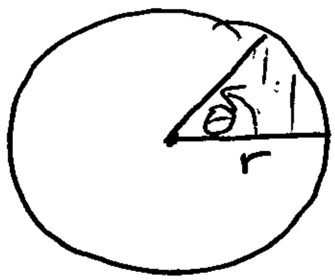
§10.4 Areas and Lengths in Polar Coordinates

HW §10.4 #1–41 odd

~~Ae. Areas Bounded~~ by

a) The Area bounded by a Polar curve $r=f(\theta)$.

Aside:



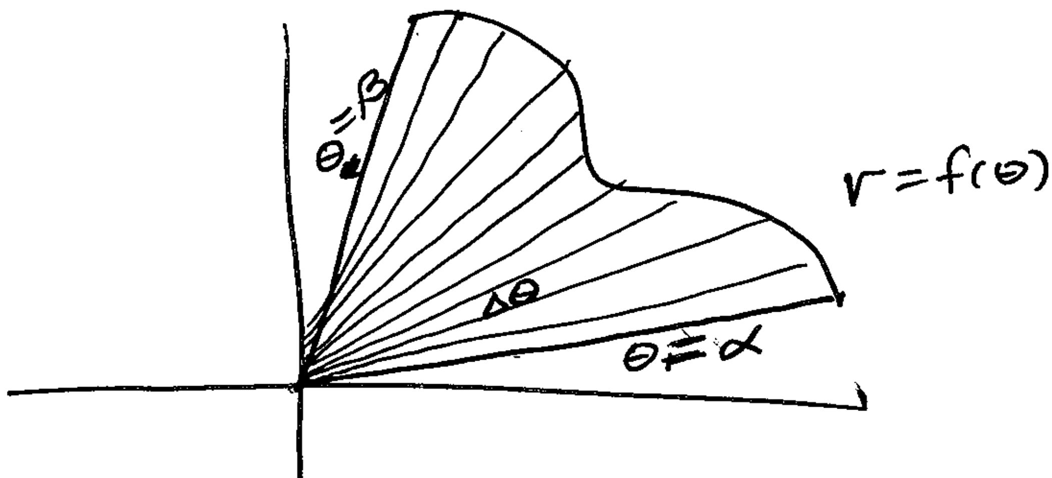
Area of sector is:

$$\frac{A}{\pi r^2}=\frac{\theta}{2\pi}$$

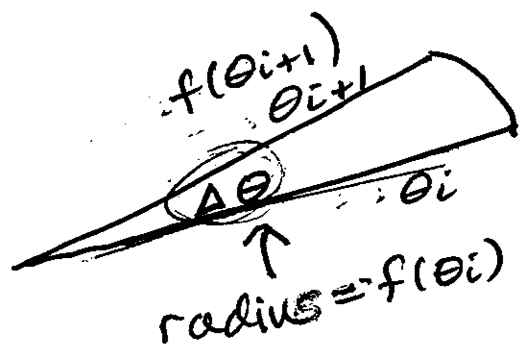
$$A=\frac{\theta}{2\cancel{\pi}}\cdot\cancel{\pi}r^2$$

Area of sector:

$$A=\frac{r^2\theta}{2}$$



Look at the ith slice



Area: $A_i = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} f(\theta_i)^2 \Delta\theta$

Add all together

$$\sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

Take ~~Find~~ the limit.

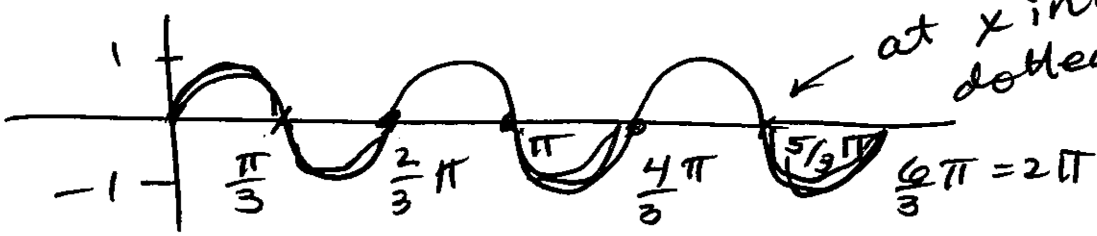
$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

$$\text{Area} = \int_\alpha^\beta \frac{1}{2} \left(f(\theta)\right)^2 d\theta$$

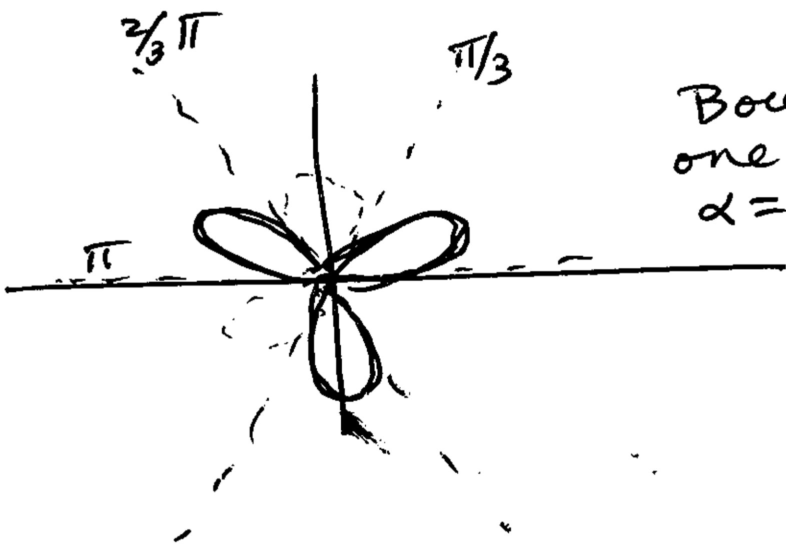
EXAMPLE Find the area of one leaf of the rose $r = \sin 3\theta$.

(a) sketch the graph.

$y = \sin 3x \qquad P = \frac{2\pi}{3}$



at x int, make dotted lines



Boundary for one loop
$\alpha = 0, \ \beta = \frac{\pi}{3}$

(b) Set up integral for area.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta$$

$$= \int_0^{\pi/3} \frac{1}{2} (\sin 3\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/3} \frac{1}{2} (1 - \cos 2(3\theta)) \, d\theta$$