§10.1 #9 a) sketch by plotting points

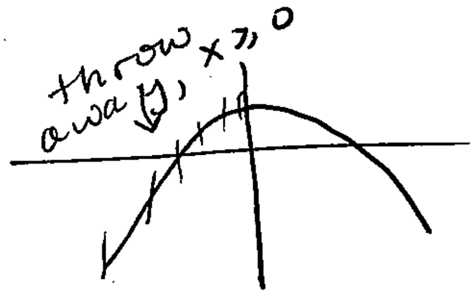
b) Eliminate the parameter.

$x = \sqrt{t}$, $y = 1 - t$

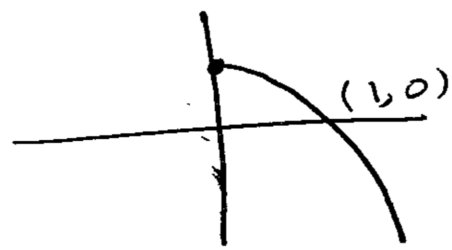
$x^2 = t$, $y = 1 - t = 1 - x^2$

$y = 1 - x^2$

$y = -x^2 + 1$



$t \geq 0$ so $x \geq 0$



§10.1 # 13 a) Eliminate the parameter
b) sketch the curve. Draw arrows to indicat diretion.

$x = \sin t$, $y = \csc t$, $0 < t < \frac{\pi}{2}$

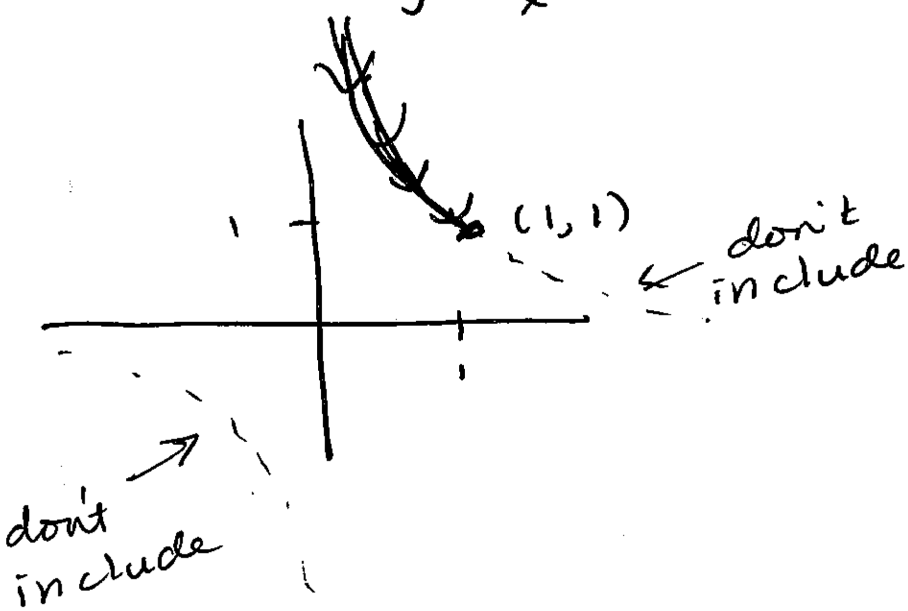
Solution

$y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$.

$y = \frac{1}{x}$.

$t = 0$, $x = \sin(0) = 0$
$y = \csc(0)$ undef

$t = \frac{\pi}{2}$ $x = \sin(\frac{\pi}{2}) = 1$
$y = \csc t = 1$



§ 10.1 #19

Describe the motion of a particle with position (x,y) as t varies in the given interval.

$$x = 3 + 2\cos t, \quad y = 1 + 2\sin t,$$

$$\frac{\pi}{2} \le t \le \frac{3}{2}\pi$$

SOLUTION

- solve for $\cos t$ & $\sin t$.

  Then use $\cos^2 t + \sin^2 t = 1$.

$x = 3 + 2\cos t$

$x - 3 = 2\cos t$

$\cos t = \frac{x-3}{2}$

$y = 1 + 2\sin t$

$y - 1 = 2\sin t$

$\sin t = \frac{y-1}{2}$

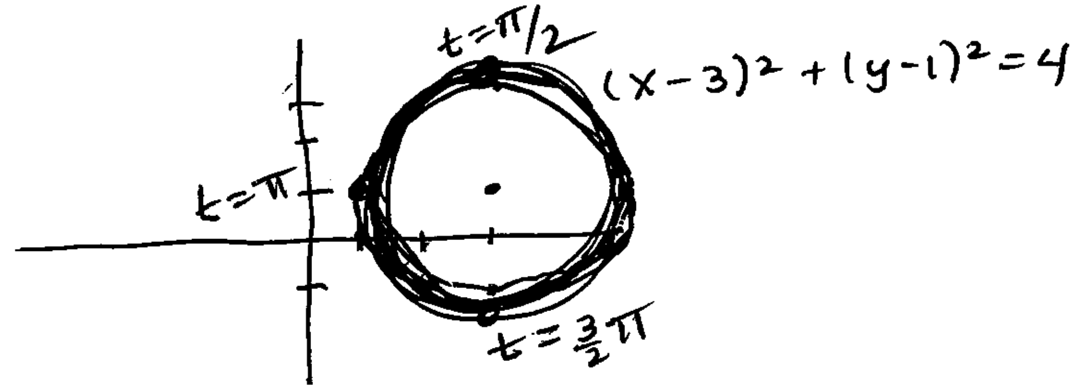
$$1 = \cos^2 t + \sin^2 t = \left(\frac{x-3}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2$$

$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{4} = 1$$

$$(x-3)^2 + (y-1)^2 = 4$$

$$(x-3)^2 + (y-1)^2 = 2^2$$

Circle center $(3,1)$, radius $r = 2$



- when $t = \frac{\pi}{2}$

$x = 3 + 2\cos\frac{\pi}{2} = 3 + 2(0) = 3$

$y = 1 + 2\sin\left(\frac{\pi}{2}\right) = 1 + 2(1) = 3$

t	x	y
$\frac{\pi}{2}$	3	3
π	~~2~~ 1	1
$\frac{3\pi}{2}$	3	-1

- when $t = \pi$

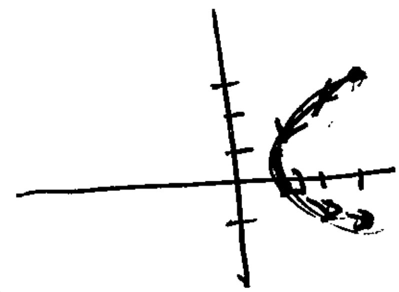
$x = 3 + 2\cos\pi = 3 + -2 = \cancel{2}\,1$

$y = 1 + 2\sin\pi = 1 + 2(0) = 1$

- when $t = \frac{3}{2}\pi$

$x = 3 + 3\cos\left(\frac{3}{2}\pi\right) = 3 + 0 = 3$

$y = 1 + 2\sin\left(\frac{3}{2}\right) = 1 + 2(-1) = -1$



The particle traces the left semicircle counterclockwise with center (3, 1), radius 2, once starting at (3, 3) and going to (3, -1)

§ 10.1 #21 Describe the motion of a particle with position (x, y) as t varies in the given interval

$$x = 5\sin t, \quad y = 2\cos t, \quad -\pi \le t \le 5\pi$$

$$\frac{x^2}{25} + \frac{y^2}{4} = \frac{(5\sin t)^2}{25} + \frac{(2\cos t)^2}{4}$$

$$= \frac{25}{25}\sin^2 t + \frac{4}{4}\cos^2 t$$

$$= \sin^2 t + \cos^2 t = 1$$

$\frac{x^2}{25} + \frac{y^2}{4} = 1$ is an ellipse

when $y=0$

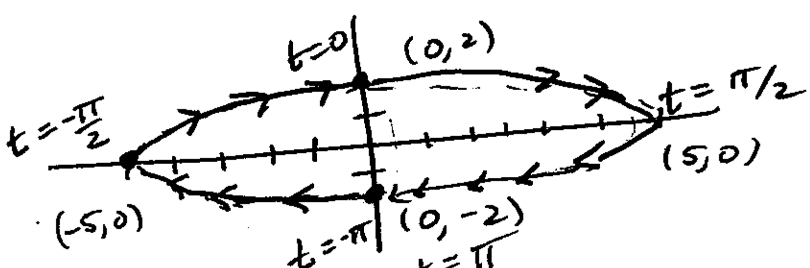
$$\frac{x^2}{25} = 1$$

$$x = \pm 5$$

when $x = 1$

$$\frac{y^2}{4} = 1$$

$$y = \pm 2$$



t	x	y
$-\pi$	0	-2
$-\frac{\pi}{2}$	-5	0
0	0	2
$\frac{\pi}{2}$	5	0
π	0	-2
5π	0	-2

The particle goes in a ~~counter~~ clockwise path ~~abou~~ around the ellipse, starting at $(0, -2)$, going around 3 times, ending at $(0, -2)$

§ 10.2 # 11 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave up?

$$x = 4 + t^2, \quad y = t^2 + t^3$$

SOLUTION $\quad x' = 2t, \quad y' = 3t^2 + 2t$

$$\frac{dy}{dx} = \frac{y'}{x'} = \cancel{3 2t} \frac{3t^2+2t}{2t}$$

$$= \frac{3t^2}{2t} + \frac{2t}{2t} = \boxed{\frac{3}{2}t + 1}$$

$$Y = \frac{dy}{dx} = \frac{3}{2}t + 1, \qquad Y' = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = \frac{dY}{dx} = \frac{Y'}{x'} = \frac{3/2}{2t} = \boxed{\frac{3}{4t}}$$

Concave up when $\quad \frac{d^2y}{dx^2} > 0$

$\frac{3}{4t} > 0 \quad$ when $t > 0$

$\boxed{\text{concave up for } t > 0}$

§ 10.2 #13 Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

For which values of t is the curve concave upword?

$x = t - e^t, \quad y = t + e^{-t}$

SOLUTION

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - e^{-t}}{1 - e^t}$$

$$= \left(\frac{1 - e^{-t}}{1 - e^t}\right)\frac{e^t}{e^t}$$

$$= \frac{e^t - e^t e^{-t}}{e^t(1 - e^t)}$$

$$= \frac{e^t - 1}{e^t(1 - e^t)} = \frac{-(1 - e^t)}{e^t(1 - e^t)}$$

$$= \frac{-1}{e^t} = -e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{(dy/dx)'}{(x)'} = \frac{(-e^{-t})'}{(t - e^t)'} = \frac{e^{-t}}{1 - e^t}$$

Concave up when $\frac{e^{-t}}{1 - e^t} > 0$.

$e^{-t} > 0$ for all t.

$1 - e^t > 0$

$1 > e^t$

$\ln 1 > \ln e^t$

$0 > t$

$\boxed{t < 0 \text{ concave up}}$

§10.2 #44 Find the exact length of the curve.

$x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$,

$0 \le t \le \pi$

SOLUTION

$L = \int ds$

$(ds)^2 = (dx)^2 + (dy)^2$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$L = \int \sqrt{(x')^2 + (y')^2}\, dt$

$x' = -3\sin t + 3\sin 3t$

$y' = 3\cos t - 3\cos 3t$

$$L = \int_0^{\pi} \sqrt{(-3\sin t + 3\sin 3t)^2 + (3\cos t - 3\cos 3t)^2}\ dt$$

$$= \int_0^{\pi} \sqrt{9\sin^2 t - 2(3)(3)\sin t \sin 3t + 9\sin^2 3t + 9\cos^2 t - 2(3)(3)\cos t\cos 3t + 9\cos^2 3t}\ dt$$

$$= \int_0^{\pi} \sqrt{9(\underbrace{\sin^2 t + \cos^2 t}_{1}) + 9(\underbrace{\sin^2 3t + \cos^2 3t}_{1}) - 18(\sin t \sin 3t + \cos t \cos 3t)}\ dt$$

$$= \int_0^{\pi} \sqrt{9 + 9 - 18(\cos(3t - t))}\ dt$$

<u>Aside</u>: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$A = 3t \quad B = t$

$$L = \int_0^{\pi} \sqrt{18 - 18\cos 2t}\, dt$$

$$= \int_0^{\pi} \sqrt{18}\sqrt{1-\cos 2t}\, dt$$

$$= 3\sqrt{2}\int_0^{\pi} \sqrt{2(\sin^2 t)}\, dt$$

$$= 3\sqrt{2}\int_0^{\pi} \sqrt{2}\, \sin t\, dt$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$= 6(-\cos t)\Big|_0^{\pi}$$

$$= -6\left[\cos\pi - \cos 0\right]$$

$$= -6\left[-1 - 1\right]$$

$$= -6(-2) = 12$$

$\sqrt{a^2} = |a|$

if $a > 0$

$\sqrt{a^2} = a$

$\sin\theta \geq 0$ on $[0, \pi]$

so $\sqrt{\sin^2\theta} = \sin\theta$ on $[0, \pi]$

§ 10.2 # 41 Find the exact length of the curve. $x = 1+3t^2,\ y = 4+2t^3,\quad 0 \le t \le 1$

SOLUTION $L = \int ds$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int \sqrt{(x')^2 + (y')^2}\, dt$$

$$x' = 6t,\quad y' = 6t^2$$

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^1 \sqrt{36t^2(1+t^2)}\, dt$$

$$= \int_0^1 6t\sqrt{1+t^2}\, dt$$

$u = 1+t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$t=0,\ u=1$

$t=1,\ u=2$

$$= \int_1^2 6 \cdot \frac{1}{2} u^{1/2}\, du$$

$$= 3\, \frac{u^{3/2}}{3/2}\Big|_1^2 = 3 \cdot \frac{2}{3}\left(2^{3/2} - 1^{3/2}\right)$$

$$= 2\,(2\sqrt{2} - 1)$$

$$= 4\sqrt{2} - 2$$

§10.2 #45 Sketch the graph and find its length.

$$x = e^t \cos t, \quad y = e^t \sin t$$

$$0 \le t \le \pi$$

<u>SOLUTION</u>: Use graphing calculator & to sketch graph.

$$L = \int ds = \int \sqrt{(x')^2 + (y')^2}\, dt$$

$$x' = e^t \cos t + e^t(-\sin t)$$

$$y' = e^t \sin t + e^t \cos t$$

$$L = \int_0^\pi \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \cos t + e^t \sin t)^2}\, dt$$

$$= \int_0^\pi \sqrt{(e^t)^2 \left[\cos^2 t \cancel{- 2\cos t \sin t} + \sin^2 t + \cos^2 t \cancel{+ 2 \sin t \cos t} + \sin^2 t\right]}\, dt$$

$$= \int_0^\pi e^t \sqrt{2}\, dt$$

$$= 2e^t \Big|_0^\pi = 2e^\pi - 2e^0 = 2e^\pi - 2$$

§10.2#61 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

$$x = a\cos^3\theta,\quad y = a\sin^3\theta,\qquad 0 \le \theta \le \frac{\pi}{2}$$

SOLUTION

$$S = \int 2\pi y\, ds = \int 2\pi y\sqrt{(x')^2+(y')^2}\, dt$$

$$x' = 3a\cos^2\theta(-\sin\theta) \qquad y' = 3a\sin^2\theta\cos\theta$$

$$= -3a\cos^2\theta\sin\theta$$

$$S = 2\pi\int_0^{\pi/2} a\sin^3\theta\sqrt{(-3a\cos^2\theta\sin\theta)^2+(3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 2a\pi\int_0^{\pi/2}\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$= 2a\pi\int_0^{\pi/2}\sin^3\theta\sqrt{9a^2\cos^2\theta\sin^2\theta\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}}\, d\theta$$

$$= 2a\pi\int_0^{\pi/2}\sin^3\theta\cdot 3a\cos\theta\sin\theta\, d\theta$$

$$= 6a^2\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$\theta = 0,\ u = \sin 0 = 0$

$\theta = \frac{\pi}{2},\ u = \sin\frac{\pi}{2} = 1$

$$= 6a^2\pi\int_0^1 u^4\, du$$

$$= 6a^2\pi\frac{u^5}{5}\Big|_0^1 = \frac{6a^2}{5}\pi$$

§10.2 #65 Find the surface area generated by rotating the given curve about the y-axis

$x = 3t^2$, $y = 2t^3$, $0 \le t \le 5$

SOLUTION $S = \int 2\pi x \, ds$

$= \int 2\pi x \sqrt{(x')^2 + (y')^2} \, dt$

$x' = 6t$, $y' = 6t^2$

$S = \int_0^5 2\pi(3t^2) \sqrt{(6t)^2 + (6t^2)^2} \, dt$

$= 6\pi \int_0^5 t^2 \sqrt{(6t)^2 [1 + \cancel{6}t^2]} \, dt$

$= 6\pi \int_0^5 t^2 \cdot 6t \sqrt{1 + t^2} \, dt$

$= 36\pi \int_0^5 t^3 \sqrt{1 + t^2} \, dt$ $u = t^2 + 1$

$du = 2t\,dt$

$= 36\pi \int_0^5 t^2 \cdot t \sqrt{1 + t^2}$ $\frac{1}{2} du = t\,dt$

$t^2 = u - 1$

$= 36\pi \cdot \frac{1}{2} \int_1^{26} (u-1) u^{1/2} \, du$ $t=0,\ u=1$

$t=5,\ u=26$

$= 18\pi \int_1^{26} (u^{3/2} - u^{1/2} \, du$

$= 18\pi \left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right]_1^{26}$

$= 36\pi \left[\frac{u^2\sqrt{u}}{5} - \frac{u\sqrt{u}}{3}\right]_1^{26}$

$= 36\pi \left[\frac{(26)^2\sqrt{26}}{5} - \frac{26\sqrt{26}}{3}\right] - 36\pi\left[\frac{1}{5} - \frac{1}{3}\right]$

$= 36\pi \cdot 26\sqrt{26} \left[\left(\frac{26}{5} - \frac{1}{3}\right)\right] - 36\pi\left(\frac{-2}{15}\right)$

§10.2 #65 continued

$$= \cancel{36}^{12}\pi \cdot 26\sqrt{26}\ \frac{73}{\cancel{15}_5} + \cancel{36}^{12}\pi\left(\frac{2}{\cancel{15}_5}\right)$$

$$= \frac{12}{5}\pi\left(26\sqrt{26}\cdot 73 + 2\right)$$

$$= \frac{24}{5}\pi\left(13\cdot 73\sqrt{26} + 1\right)$$

$$= \frac{24}{5}\pi\left(949\sqrt{26} + 1\right)$$

$$\begin{array}{r} 26 \\ \times\ 3 \\ \hline 78 \\ -\ 5 \\ \hline 73 \end{array}$$

$$\begin{array}{r} 73 \\ \times\ 13 \\ \hline 219 \\ 730 \\ \hline 949 \end{array}$$