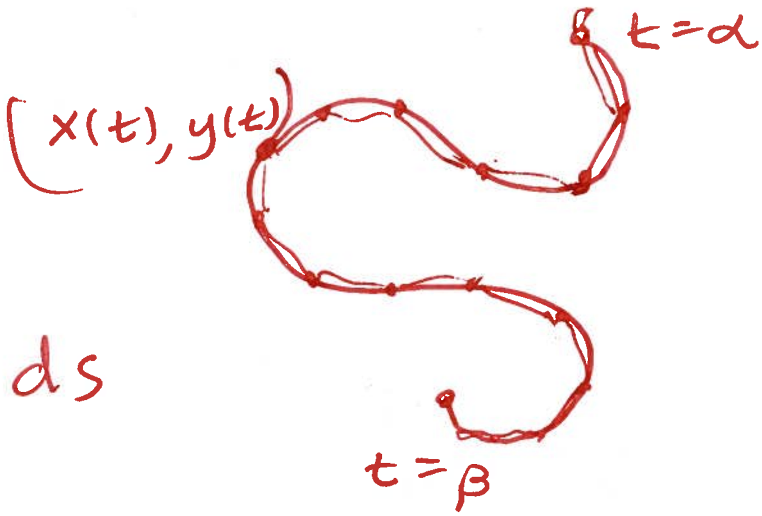
§10.2 Continued

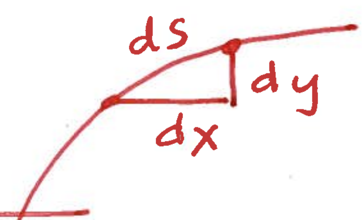
Arc Length



$$L = \int ds$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$



$$ds = \sqrt{\left((dx)^2 + (dy)^2\right)\frac{(dt)^2}{(dt)^2}}$$

$$ds = \sqrt{\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2}}\ dt$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt, \qquad \alpha \le t \le \beta$$

where the curve is traversed exactly once from $t = \alpha$ to $t = \beta$.

Example: One arch of a cycloid is given by $0 \leq \theta \leq 2\pi$.

$x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$

Find the length of one arch.

SOLUTION

$$L = \int ds = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = r(1-\cos\theta), \quad \frac{dy}{d\theta} = r(0-(-\sin\theta)) = r\sin\theta$$

$$0 \leq \theta \leq 2\pi$$

$$L = \int_0^{2\pi} \sqrt{[r(1-\cos\theta)]^2 + [r\sin\theta]^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1-2\cos\theta+\cos^2\theta) + r^2\sin^2\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{1-2\cos\theta + \underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2-2\cos\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

FORMULA $\quad \sin^2 x = \dfrac{1-\cos 2x}{2}$

$\theta = 2x$

$\frac{\theta}{2} = x$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

$$2\sin^2\left(\frac{\theta}{2}\right) = 1-\cos\theta$$

$$L = r\int_0^{2\pi} \sqrt{2\cdot 2\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

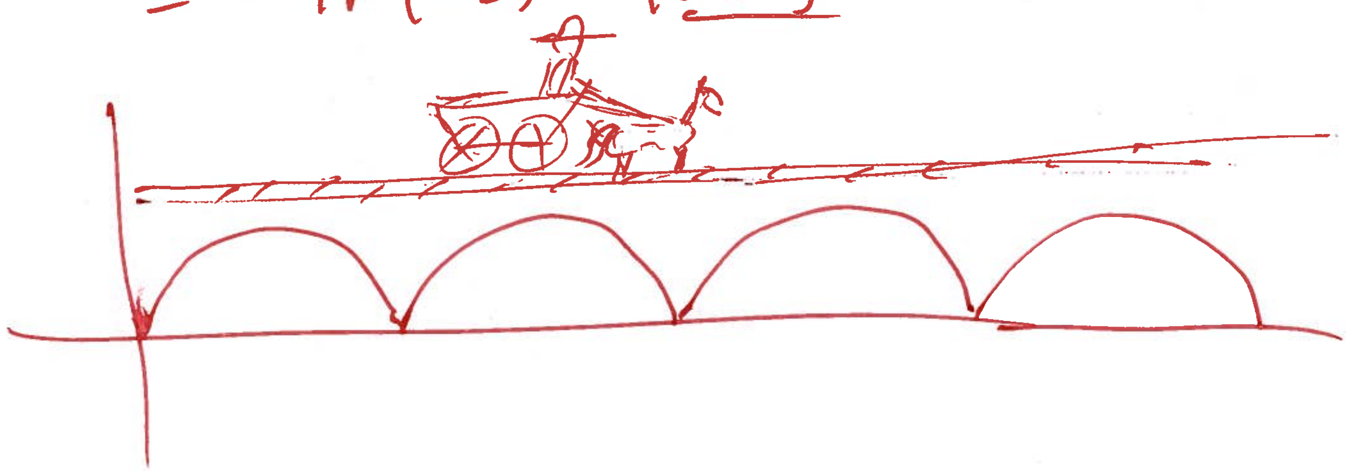
$$= 2r\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 2r\left[-\cos\left(\frac{\theta}{2}\right)\cdot 2\right]_0^{2\pi}$$

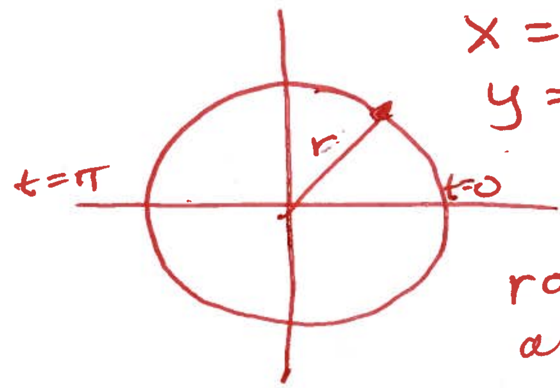
$$= -4r\left[\cos\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) - \cos\left(\frac{0}{2}\right)\right]$$

$$= -4r\left[-1-1\right] = \cancel{-4\pi(-2) = 8\pi}$$

$$= -4r(-2) = \boxed{8r}$$



Example: Show that the surface area of a sphere of radius r is $S = 4\pi r^2$.



$x = r\cos t$
$y = r\sin t$

rotate top about x-axis

SOLUTION

about x-axis

$S = \int 2\pi y \, ds$

$= \int 2\pi y \sqrt{(x')^2 + (y')^2} \, dt$

$0 \le t \le \pi$

$x' = -r\sin t$
$y' = r\cos t$

$= \int_0^{\pi} 2\pi \underbrace{(r\sin t)}_{=y} \sqrt{(-r\sin t)^2 + (r\cos t)^2} \, dt$

$= \int_0^{\pi} 2\pi r \sin t \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} \, dt$

$= \int_0^{\pi} 2\pi r \sin t \sqrt{r^2 (\underbrace{\sin^2 t + \cos^2 t}_{1})} \, dt$

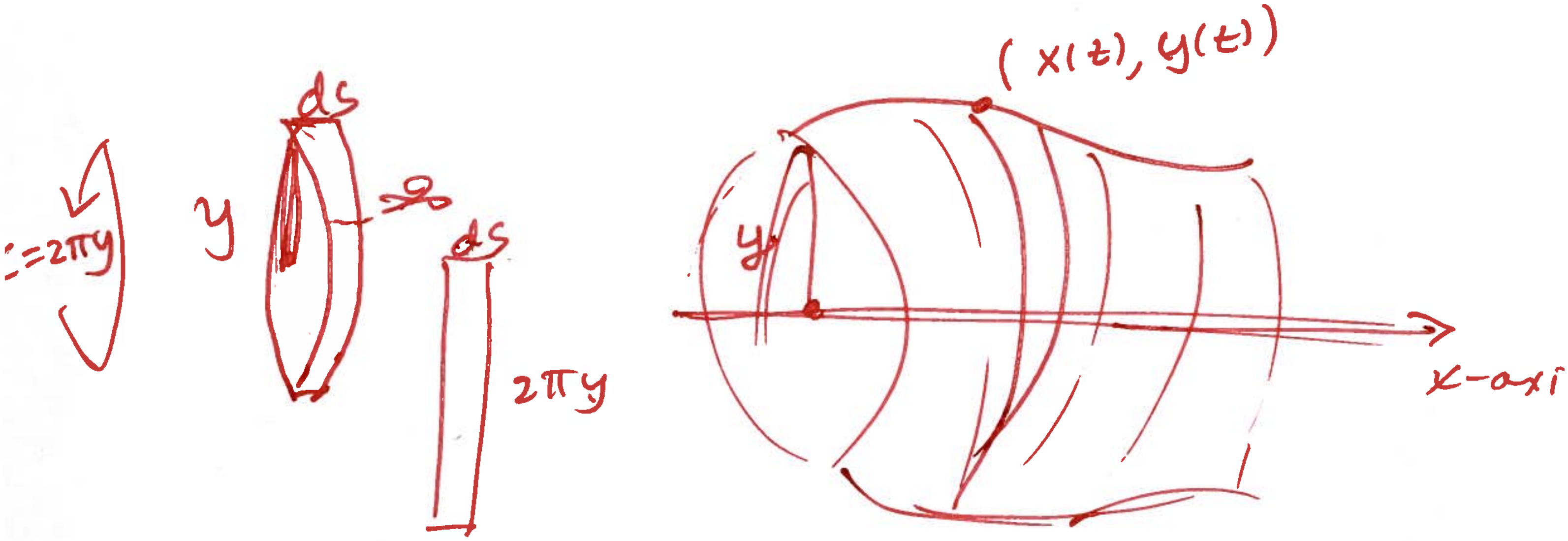
$= \int_0^{\pi} 2\pi r \cdot r \sin t \, dt$

$= 2\pi r^2 \int_0^{\pi} \sin t \, dt = 2\pi r^2 \left[-\cos t \right]_0^{\pi}$

$= -2\pi r^2 [\cos \pi - \cos 0] = -2\pi r^2 [-1 - 1]$

$= -2\pi r^2 (-2) = 4\pi r^2 \quad \square$

Surface Area



Surface Area, • rotated about x-axis

$$S = \int 2\pi y \, ds$$

• rotated about the y-axis

$$S = \int 2\pi x \, ds$$

• $(ds)^2 = (dx)^2 + (dy)^2$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$