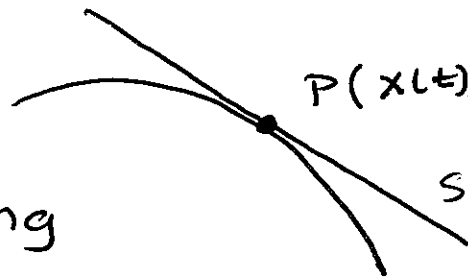
§10.2 Calculus with Parametric Curves.

HW §10.2 #1–20, 41–44, 59–61

Tangents

$P(x(t), y(t))$



Slope of tangent line is $\frac{dy}{dx}$

We have, using the chain rule

$$\cancel{\frac{dy}{dx} = dx} \qquad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

divide through by $\frac{dx}{dt}$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

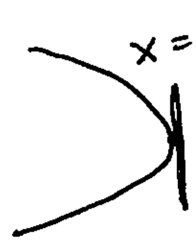
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}, \qquad \frac{dx}{dt} \neq 0$$

We also have

$$\frac{dx}{dy} = \frac{x'(t)}{y'(t)}$$

$x = g(y)$



At a vertical tangent line $\frac{dx}{dy} = 0$

Let's find a formula for the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \cancel{\frac{d}{dx}}$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Here we assume $Y = \frac{dy}{dx}$ is a function of x, so

$$\frac{dY}{dx} = \frac{\frac{d}{dt}Y}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example: Find dy/dx and d^2y/dx^2.

#12 $\quad x = t^3 - 12t, \quad y = t^2 - 1$

$$\frac{dy}{dt} = 2t, \qquad \frac{dx}{dt} = 3t^2 - 12$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-12}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cancel{dt}\ \frac{d}{dt}\left(\frac{2t}{3t^2-12}\right)^{\text{(I)}}}{3t^2-12}$$

(I) $\quad \frac{d}{dt}\left(\frac{2t}{3t^2-12}\right) = \frac{(2t)'(3t^2-12) - (2t)(3t^2-12)'}{(3t^2-12)^2}$

$$= \frac{2(3t^2-12) - 2t(6t)}{(3t^2-12)^2}$$

$\cancel{\frac{d^2y}{dx^2}}$

$$= \frac{6t^2 - 24 - 12t^2}{(3t^2-12)^2}$$

$$= \frac{-6t^2 - 24}{(3t^2-12)^2}$$

$$= \frac{-6(t^2+4)}{3^2(t^2-4)^2}$$

$$= -\frac{2(t^2+4)}{3(t^2-4)^2}$$

$$\frac{dy}{dx} = \frac{\left(\frac{-2(t^2+4)}{3(t^2-4)^2}\right)}{3t^2-12} = \frac{-2(t^2+4)}{3(t^2-4)^2} \cdot \frac{1}{(3t^2-12)}$$

$$\frac{d^2y}{dx^2} = \frac{-2(t^2+4)}{9(t^2-4)^3}$$

~~For wit~~: For which values of t is the curve concave up?

when $\frac{d^2y}{dx^2} > 0$

Solve $\frac{-2(t^2+4)}{9(t^2-4)^3} > 0$

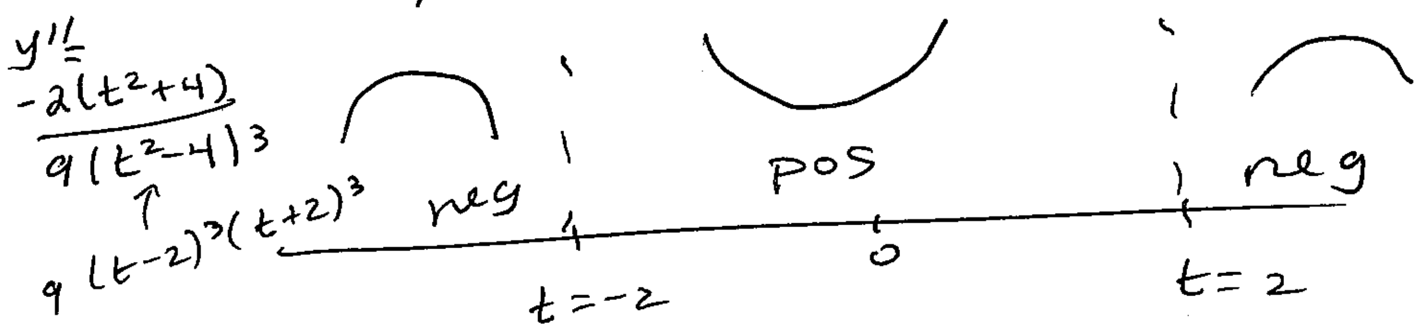
zeros $\quad \dfrac{-2(t^2+4)}{9(t^2-4)^3} = 0 \qquad$ no zeros

$$-2(t^2+4) = 0$$
$$t^2+4 = 0$$
$$t^2 = -4$$

undef $\quad$ when denom is zero.

$$9(t^2-4)^3 = 0$$
$$t^2-4 = 0$$
$$t^2 = 4$$
$$t = \pm 2$$

Incr/Decr Chart



Test Numbers $\quad t=0$

$$y'' = \frac{-2(4)}{9(-4)^3} = \frac{-8}{9(-64)} \quad \text{pos}$$

$t=3$

$$y'' = \frac{-2(3^2+4)}{9(3^2-4)^3} \quad \text{neg}$$

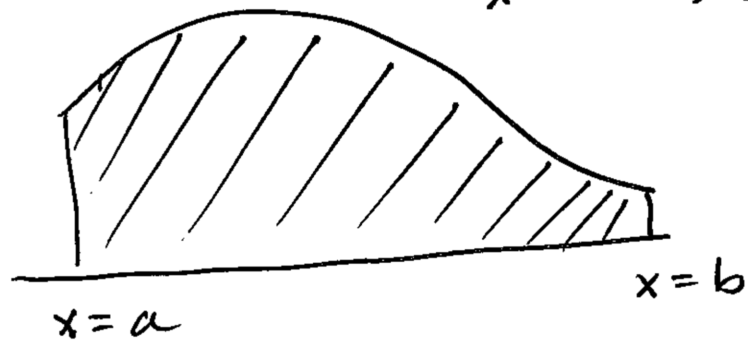
$t=-3 \qquad y'' = \dfrac{-2((-3)^2+4)}{9((-3)^2-4)^3} \quad$ neg

Answer: $\quad -2 < t < 2$

i.e. $\quad (-2, 2)$

Areas

$x = f(t),\ y = g(t)$



The area under the curve.

$$\int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

substitution

$x = f(t) \qquad y = g(t)$

$dx = f'(t)\,dt$

when $x = a$, say $t = \alpha$

when $x = b$, say $t = \beta$

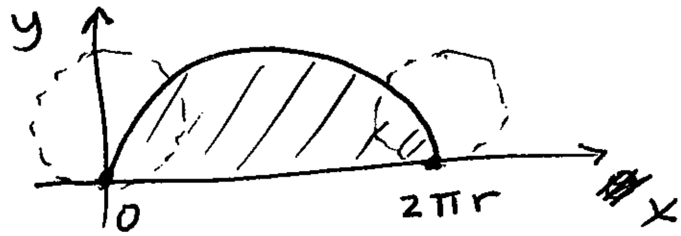
i.e. $f(\alpha) = a$, $\cancel{g(\beta)}$ $f(\beta) = b$

$$A = \int_\alpha^\beta g(t)f'(t)\,dt$$

where $x = f(t)$, $y = g(t)$.

Example: Find the area under one arch of the cycloid.

$$x = r(\theta - \sin\theta), \quad y = r(1-\cos\theta)$$



$$A = \int_0^{2\pi r} y\,dx = \cancel{\int r(1-\cos\theta} = \cancel{\int g(\tfrac{d\theta}{d\theta})}\,d\theta = \int y\left(\frac{dx}{d\theta}\right)$$

$$= \int_\alpha^\beta g(\theta) f'(\theta)\,d\theta$$

$$\cancel{\frac{dx}{d\theta} = r f\theta}$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} r(\theta - \sin\theta)$$

$$= r(1-\cos\theta)$$

when $x=0$

$r(\theta - \sin\theta) = 0$

$\theta - \sin\theta = 0$

$\sin\theta = \theta$

$\boxed{\theta = 0}$

when $x = 2\pi r$

$r(\theta - \sin\theta) = 2\pi r$

$\theta - \sin\theta = 2\pi$

$\theta = 2\pi$

$(2\pi) - \sin(2\pi)$

$= 2\pi - 0$

$= 2\pi$

$$A = \int_0^{2\pi} \underbrace{r(1-\cos\theta)}_{y}\ \underbrace{r(1-\cos\theta)}_{\frac{dx}{d\theta}}\,d\theta$$

$$A = r^2 \int_0^{2\pi} (1-\cos\theta)^2\,d\theta$$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \cos^2\theta\right] d\theta$$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \frac{1}{2}\left(1 + \cos 2\theta\right)\right] d\theta$$

$$= r^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= r^2 \left[\frac{3}{\cancel{2}}(\cancel{2}\pi) - 2\overbrace{\sin(2\pi)}^{=0} + \frac{1}{4}\overbrace{\sin(2\cdot 2\pi)}^{=0}\right]$$

$$- r^2 \left[\frac{3}{2}(0) - 2\underbrace{\sin(0)}_{=0} + \frac{1}{4}\underbrace{\sin(2\cdot 0)}_{=0}\right]$$

$$= 3\pi r^2$$

§9.5 Linear DE's

#12)

$$x\frac{dy}{dx} - 4y = x^4 e^x$$

$$\frac{dy}{dx} - \frac{4}{x}y = x^3 e^x$$

Integrating factor

$$I = e^{\int P(x)dx}$$
$$= e^{\int -\frac{4}{x}dx}$$
$$= e^{-4\ln x}$$
$$= e^{\ln x^{-4}}$$
$$= x^{-4} = \frac{1}{x^4}$$

Multiply through by $I = \frac{1}{x^4}$

$$\frac{1}{x^4}\frac{dy}{dx} - \frac{4}{x^5}y = \frac{e^x}{x}$$

$$\left(\frac{1}{x^4}y\right)' = \frac{e^x}{x}$$

$$\frac{1}{x^4}y = \int \frac{e^x}{x}dx + C$$

$$\boxed{y = x^4\left[\int \frac{e^x}{x}dx + C\right]}$$

$\int \frac{e^x}{x}\,dx$

Int. by Parts

$u = \frac{1}{x} = x^{-1}$ $\quad dv = e^x dx$

$du = -1 \cdot x^{-2} dx$ $\quad v = e^x$

$du = \frac{-1}{x^2} dx$

$= uv - \int v\,du = \frac{e^x}{x} + \int e^x \left(\cancel{-}\frac{1}{x^2}\right) dx$

$\int \frac{e^x}{x}\,dx$

$u = e^x$ $\quad dv = x^{-1} dx$

$du = e^x dx$ $\quad v = \ln|x|$

$= e^x \ln|x| - \int e^x \ln|x|\,dx$

Quiz Tomorrow
§9.3, 9.5

§9.3 Separable DE's

#16 $xy' + y = y^2, \quad y(1) = -1$

$$x\frac{dy}{dx} + y = y^2$$

$$x\frac{dy}{dx} = y^2 - y$$

$$\frac{dy}{dx} = \frac{y^2-y}{x}$$

$$\int^{\text{I}} \frac{1}{y^2-y}\,dy = \int^{\text{II}} \frac{1}{x}\,dx$$

I $\int \frac{1}{y^2-y}\,dy$ Partial Fractions

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

$1 = A(y-1) + By$

$y = 1$

$1 = B\cancel{y}1, \quad B = 1$

$y = 0$

$1 = A(0-1)$

$A = -1$

$$\int\left(\frac{-1}{y} + \frac{1}{y-1}\right)dy = -\ln|y| + \ln|y-1|$$

II $\int \frac{1}{x}\,dx = \ln|x|$

$$-\ln|y| + \ln|y-1| = \ln|x| + C$$

$$\ln\left|\frac{y-1}{y}\right| = \ln|x| + C$$

$$e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|x| + C}$$

$$\left|\frac{y-1}{y}\right| = e^{\ln|x|} \cdot e^{C}$$

$$\left|\frac{y-1}{y}\right| = e^{C}|x|$$

$$\frac{y-1}{y} = \pm e^{C}x$$

Let $k = \pm e^{C}$

$$\frac{y-1}{y} = kx$$

Solve for k
$y(1) = -1$
when $x = 1$, $y = -1$

$$\frac{(-1)-1}{(-1)} = k(1)$$

$$\frac{-2}{-1} = k, \quad k = 2$$

$$\frac{y-1}{y} = 2x$$

$$y - 1 = 2xy$$

$$y - 2xy = 1$$

$$y(1-2x) = 1$$

$$\boxed{y = \frac{1}{1-2x}}$$