§11.10 #50 Evaluate the indefinite integral as a ~~power~~ infinite series.

Use Table 1.

$$\int \arctan(x^2)\,dx$$

SOLUTION

Use $\displaystyle \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad |x|<1, \quad R=1$$

$$\arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}, \quad R=1 \qquad |x^2|<1, \quad |x|<1$$

$$\arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}, \quad R=1$$

$$\int \arctan(x^2)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\int x^{4n+2}\,dx\right)$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{x^{4n+3}}{4n+3}, \quad R=1$$

§11.10 #31 $f(x) = e^x + e^{2x}$

Use Table 1 to find the Maclaurin Series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{(2x)} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$= 1 + 2x + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} + \frac{2^4 x^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \qquad R = \infty$$

$$e^x + e^{2x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} + \frac{2^n}{n!} \right) x^n, \quad R = \infty$$

$$= \sum_{n=0}^{\infty} \frac{2^n + 1}{n!} x^n, \quad R = \infty$$

§11.10 #35 Find the Maclaurin Series using Table 1

$$f(x) = \frac{x}{\sqrt{4+x^2}}$$

$$= x\ (4+x^2)^{-1/2}$$

$$= x \cdot (4^{-1/2}) \left(1 + \frac{x^2}{4}\right)^{-1/2}$$

$$= \frac{x}{2}\left(1 + \frac{x^2}{4}\right)^{-1/2}$$

Use $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ $\quad R=1$, $|x|<1$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

$$= \frac{x}{2}\left(1+\left(\frac{x^2}{4}\right)\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x^2}{4}\right)^n$$

$k = -1/2$

$\left|\frac{x^2}{4}\right| < 1$

$x^2 < 4$

$|x| < 2$

$R = 2$

$$= \frac{x}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n}}{\underbrace{4^n}_{=2^{2n}}}$$

$$= \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+1}}{2^{2n+1}}, \quad R=2$$

$n=0 \quad \binom{-1/2}{0} = 1$

$n=1 \quad \binom{-1/2}{1} = -1/2$

$n=2 \quad \binom{-1/2}{2} = \dfrac{\left(-\frac{1}{2}\right)\left(-3/2\right)}{2!} = \dfrac{1\cdot 3}{2^2\cdot 2!}$

$n=3 \quad \binom{-1/2}{3} = \dfrac{(-1/2)(-3/2)(-5/2)}{3!} = -\dfrac{1\cdot 3\cdot 5}{2^3\cdot 3!}$

$n=4 \quad \binom{-1/2}{4} = \dfrac{(-1/2)(-3/2)(-5/2)(-7/2)}{4!} = \dfrac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}$

$$\binom{-1/2}{n} = \frac{(-1)^n(1\cdot 3\cdot 5\cdots(2n-1))}{2^n\, n!}, \quad n \geq 1$$

$$\frac{x}{\sqrt{4+x^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n+1}}{2^{2n+1}}, \quad R=2$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n(1\cdot 3\cdot 5\cdots(2n-1))}{2^n\, n!\cdot 2^{2n+1}}\, x^{2n+1}, \quad R=2$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5\cdots(2n-1)}{2^{3n+1}\, n!}\, x^{2n+1}, \quad R=2$$

§11.10 #25 ~~Use~~ Find the Maclaurin Series

$f(x) = \sqrt{1+x}$ $\qquad f(0) = 1$

$= (1+x)^{1/2}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$ $\qquad f''(0) = \frac{-1}{2^2}$

$f^{(3)}(x) = \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(1+x)^{-5/2}$ $\qquad f^{(3)}(0) = \frac{1\cdot 3}{2^3}$

$f^{(4)}(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(1+x)^{-7/2}$ $\qquad f^{(4)}(0) = -\frac{1\cdot 3\cdot 5}{2^4}$

$$\underset{n\geq 2}{f^{(n)}(0)} = \frac{(-1)^{n+1}(1\cdot 3\cdot 5\cdots 2n}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1\cdot 3\cdot 5\cdots(2n-3)}{2^n\, n!} x^n$$

Radius ~~conv~~ con.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}\, 1\cdot 3\cdot 5\cdots(2n-3)\cdot \overbrace{(2(n+1)-3)}^{2n-1}\; x^{n+1}}{2^{n+1}(n+1)!} \cdot \frac{2^n\, n!}{(-1)^{n+1}\, 1\cdot 3\cdots(2n-3)} \right|$$

$$= \lim \underbrace{\frac{(2n-1)}{n+1}}_{\to 2} \frac{1}{2}|x| = 2\cdot\frac{1}{2}|x| = |x| < 1$$

$$\boxed{R=1}$$

$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\, 1\cdot 3\cdot 5 \cdots (2n-3)\, x^n}{2^n\, n!}$$

$R = 1$

§11.9 #15 Find a power series representation for the function and determine its radius of convergence.

$f(x) = \ln(5-x)$

SOLUTION Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$

$$\frac{1}{5-x} = \frac{1}{5}\cdot\frac{1}{1-\left(\frac{x}{5}\right)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n, \quad \left|\frac{x}{5}\right|<1$$

$|x|<5$

$R=5$

Integrate both sides

$$\int \frac{1}{5-x}\,dx = C + \int\left(\frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n}\right)dx$$

$$-\ln|5-x| = C + \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} \qquad R=5$$

$$\ln(5-x) = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)}$$

let $x=0$

$$\ln(5-0) = C - \sum_{n=0}^{\infty}\frac{0^{n+1}}{5^{n+1}(n+1)}$$

$\ln 5 = C$

$$\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)}$$

$$\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{5^n \cdot n}, \quad R=5$$

$$\frac{1}{\sqrt{x}} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5 \cdots (2n-1)}{n!\, 2^n \cdot 3^{2n+1}} (x-9)^n$$

Radius of Convergence:

$$\lim_{n\to\infty} \left| \left(\frac{(-1)^{n+1}\, 1\cdot 3\cdot 5 \cdots (2n-1)(2(n+1)-1)\,(x-9)^{n+1}}{(n+1)!\, 2^{n+1} \cdot 3^{2(n+1)+1}} \right) \cdot \frac{n!\, 2^n\, 3^{2n+1}}{(-1)^n\, 1\cdot 3\cdot 5 \cdots (2n-1)\,(x-9)^n} \right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{1\cdot 3\cdot 5 \cdots (2n-1)} \cdot \frac{2^n}{2^{n+1}} \frac{3^{2n+1}}{3^{2n+3}} \left| \frac{(x-9)^{n+1}}{(x-9)^n} \right|$$

$$\lim_{n\to\infty} \left(\frac{1}{n+1}\right)(2n+1) \cdot \frac{1}{2} \cdot \frac{1}{3^2} \,|x-9|$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{2n+1}{n+1}\right)}_{2} \cdot \frac{1}{18}\,|x-9| = \frac{2}{18}|x-9|$$

$$= \frac{|x-9|}{9} < 1$$

$$|x-9| < 9$$

$$\boxed{R = 9}$$

§ 11.10 # 19 Find the Taylor's Series.

$$f(x) = \frac{1}{\sqrt{x}}, \quad a = 9$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$f(x) = x^{-1/2}$$

$$f'(x) = -\frac{1}{2} x^{-3/2}$$

$$f''(x) = -\frac{1}{2} \cdot -\frac{3}{2} x^{-5/2} = \frac{1 \cdot 3}{2^2} x^{-5/2}$$

$$f^{(3)}(x) = -\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2} x^{-7/2} = -\frac{1 \cdot 3 \cdot 5}{2^3} x^{-7/2}$$

$$f^{(4)}(x) = -\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2} \cdot -\frac{7}{2} x^{-9/2} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4} x^{-9/2}$$

$\vdots$

$$f^{(n)}(x) = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^{-\frac{(2n+1)}{2}}$$

$$f^{(n)}(9) = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \frac{1}{(9)^{\frac{2n+1}{2}}}$$

$$= \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \frac{1}{(\sqrt{9})^{2n+1}}$$

$$f^{(n)}(a) = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot 3^{2n+1}}$$

§11.6 #9 $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(1.1)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(-1)^n (1.1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{(1.1)^{n+1}}{(1.1)^n} \cdot \underbrace{\frac{n^4}{(n+1)^4}}_{\to 1}$$

$$= \lim_{n\to\infty} (1.1) \Big(\underbrace{\frac{n}{n+1}}_{\to 1} \Big)^4 = 1.1 > 0$$

Divergent.