§11.9 #12 $\qquad f(x) = \dfrac{x+2}{2x^2-x-1}$

$$\frac{x+2}{2x^2-x-1} = \frac{x+2}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$x+2 = A(x-1) + B(2x+1)$$

$$x = 1$$

$$3 = 3B, \quad B = 1$$

$$x = \frac{-1}{2}$$

$$\frac{3}{2} = \frac{-3}{2}A; \quad A = -1$$

$$= \overset{\text{I}}{\frac{-1}{2x+1}} + \overset{\text{II}}{\frac{1}{x-1}} \qquad \text{Use } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| <$$

I $\quad \dfrac{-1}{2x+1} = \dfrac{-1}{1+2x} = \dfrac{-1}{1-(-2x)} = -\displaystyle\sum_{n=0}^{\infty}(-2x)^n$

$$= -\sum_{n=0}^{\infty}(-1)^n 2^n x^n \qquad |-2x| < 1$$

$$2|x| < 1$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1} 2^n x^n \qquad |x| < \frac{1}{2}$$

II $\quad \dfrac{1}{x-1} = -\dfrac{1}{1-x} = -\displaystyle\sum_{n=0}^{\infty} x^n, \qquad |x| < 1$

$$\frac{-1}{2x+1} + \frac{1}{x-1} = \sum_{n=0}^{\infty}(-1)^{n+1} 2^n x^n + \sum_{n=0}^{\infty}(-1) x^n, \quad |x| < \frac{1}{2}$$

$$= \sum_{n=0}^{\infty}\left[(-1)^{n+1}2^n + -1\right] x^n, \quad |x| < \frac{1}{2}$$

$$R = \frac{1}{2}, \quad \left(-\frac{1}{2}, \frac{1}{2}\right)$$

§11.10 # 12 $f(x) = \cosh x$

Find the Maclaurin Series

Find R.

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = \cosh x$

$f'(x) = \sinh x$

$f''(x) = \cosh x$

$f^{(3)}(x) = \sinh x$

$f^{(4)}(x) = \cosh x$

$f(0) = \cosh(0) = \frac{e^0 + e^{-0}}{2} = 1$

$f'(0) = \sinh x = \frac{e^0 - e^{-0}}{2} = 0$

$f''(0) = \cancel{\cos}\cosh 0 = 1$

$f'''(0) = \sinh(0) = 0$

$f^{(2k)}(0) = 1$, K ~~ax~~ pos integer

$f^{(2k+1)}(0) = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(III)}(0)}{3!}x^3 \cdots$$

$$= 1 + \frac{1}{2!}x^2 + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Find R

$$\lim_{n\to\infty} \left| \frac{X^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{X^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{X^{2n+2}}{X^{2n}} \right| \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} |X^2| \frac{\cancel{(2n)!}}{\underbrace{(2n+2)}_{\to\infty}\underbrace{(2n+1)}_{\to\infty}\cancel{(2n)!}} = 0 < 1$$

$$\boxed{R = \infty}$$

§ 11.10 #16 $f(x) = \frac{1}{x}$, $a = -3$

Find the Taylor Series of f around a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$f(x) = x^{-1}$ $\qquad f(-3) = -\frac{1}{3}$

$f'(x) = -x^{-2} = -\frac{1}{x^2}$ $\qquad f'(-3) = -\frac{1}{3^2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$ $\qquad f''(-3) = -\frac{2}{3^3}$

$f'''(x) = -3\cdot 2\cdot 1 x^{-4} = \frac{-3!}{x^4}$ $\qquad f'''(-3) = -\frac{3!}{3^4}$

$f^{(4)}(x) = 4\cdot 3\cdot 2\cdot 1\cdot x^{5} = \frac{4!}{x^5}$ $\qquad f^{(4)}(-3) = \frac{-4!}{3^5}$

$f^{(n)}(-3) = \frac{-n!}{3^{n+1}}$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{-n!}{3^{n+1}\, n!} (x - -3)^n$$

$$= \sum_{n=0}^{\infty} -\frac{(x+3)^n}{3^{n+1}}$$

Find R

$$\lim_{n\to\infty}\left| -\frac{(x+3)^{n+1}}{3^{(n+1)+1}} \cdot \frac{3^{n+1}}{-(x+3)^{n}}\right|$$

$$= \lim_{n\to\infty} \frac{|x+3|}{3} = \frac{|x+3|}{3} < 1$$

$$|x+3| < 3 \quad \boxed{R = 3}$$

~~Another~~

§11.10 #26 Use the binomial series to expand the function as a power series. State the radius of convergence.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad R=1$$

$$\frac{1}{(1+x)^4} = (1+x)^{-4}, \qquad k=-4$$

$$= \sum_{n=0}^{\infty} \binom{-4}{n} x^n$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

$$= \binom{-4}{0} + \binom{-4}{1} x + \binom{-4}{2} x^2 + \binom{-4}{3} x^3 + \binom{-4}{4} x^4 + \cdots$$

$$= 1 + \frac{-4}{1!} x + \frac{(-4)(-5)}{2!} x^2 + \frac{(-4)(-5)(-6)}{3!} x^3 + \frac{(-4)(-5)(-6)(-7)}{4!} x^4 + \frac{(-4)(-5)(-6)(-7)(-8)}{5!} x^5 + \cdots$$

$$= \underset{n=0}{\frac{1\cdot2\cdot3}{6}} \; \underset{n=1}{-\frac{1\cdot2\cdot3\cdot4}{6}} x + \underset{n=2}{\frac{1\cdot2\cdot3\cdot4\cdot5}{6\,2!}} x^2 - \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6}{6\,3!} x^3 + \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}{6\,4!} x^4 - \cdots$$

$$= \underset{n=0}{\frac{3!}{6\cdot0!}} - \underset{n=1}{\frac{4!\,x}{6\cdot1!}} + \underset{n=2}{\frac{5!}{6\cdot2!}} x^2 - \frac{6!}{6\cdot3!} x^3 + \frac{7!}{6\cdot4!} x^4 - \frac{8!}{6\cdot5!} x^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+3)!}{6 \cdot n!} x^n$$

$$\frac{(n+3)!}{n!} = \frac{(n+3)(n+2)(n+1)\,n!}{n!}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n (n+3)(n+2)(n+1)}{6} x^n, \quad R = 1}$$

OR

$$\cancel{\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+3)}{2!} x^n}$$

§ 11.10 #. 28 Find a power series using a binomial series.

$$(1-x)^{2/3}$$

SOLUTION

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad R = 1$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad \binom{k}{0} = 1$$

$$(1-x)^{2/3} = (1+(-x))^{2/3} \qquad k = 2/3$$

$$= \sum_{n=0}^{\infty} \binom{2/3}{n} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{2/3}{n} x^n$$

$$|-x| < 1 \qquad R = 1$$
$$|x| < 1$$

$$= 1 - \frac{2}{3}x + \frac{(2/3)(-1/3)}{2!} x^2 - \frac{(2/3)(-1/3)(-4/3)}{3!} x^3 + \frac{(2/3)(-1/3)(-4/3)(-7/3)}{4!} x^4$$

$$= 1 - \frac{2}{3}x - \underbrace{\frac{2\cdot 1}{3^2 \cdot 2!} x^2}_{n=2} - \underbrace{\frac{2(1\cdot 4)}{3^3\cdot 3!} x^3}_{n=3} - \underbrace{\frac{2(1\cdot 4\cdot 7)}{3^4 4!} x^4}_{n=4} - \underbrace{\frac{2\cdot(1\cdot 4\cdot 7\cdot 10)}{3^5\cdot 5!} x^5}_{n=5}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{2\cdot(1\cdot 4\cdot 7\cdot 10\cdots(3n-5))}{3^n n!} x^n$$

$$R = 1$$

$$= \boxed{1 - \frac{2}{3}x - \sum_{n=2}^{\infty} \frac{2(1\cdot 4\cdot 7\cdot 10\cdots(3n-5))}{3^n n!} x^n}$$

$$\cancel{= 1 - \tfrac{2}{3}} \quad = 1 + \sum_{n=0}^{\infty} \frac{(-5)(-2)(1)(4)(7)\cdots(3n-5)\,x^n}{(-5)\,3^n\,n!}$$

$$\boxed{(1-x)^{2/3} = \sum_{n=0}^{\infty} \frac{(-5)(-2)(1)(4)(7)\cdots(3n-5)\,x^n}{(-5)\,3^n\,n!} \qquad R=1}$$

$$\sum_{n=0}^{\infty} 3n-5 = (-5)+(-2)+(1)+4+7\cdots$$

$$\prod_{k=0}^{n} (3k-5) = (-5)(-2)(1)(4)(7)\cdots(3n-5)$$

§11.10 #32 $f(x) = e^x + 2e^{-x}$

Find the Maclaurin Series using Table 1 as a starting point.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad R = \infty$$

$$2e^{-x} = 2\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}, \qquad R = \infty$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}, \qquad R = \infty$$

$$e^x + 2e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 2}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{n!} + \frac{(-1)^n \cdot 2}{n!} \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{2(-1)^n + 1}{n!} x^n, \qquad R = \infty$$

§11.10 #48 $\displaystyle\int \frac{e^x - 1}{x}\,dx$

Evaluate the indefinite integral as an infinite series.

SOLUTION

$$R = \infty$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \cdots\right) - 1$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^5}{5!} + \cdots$$

$$\int \frac{e^x - 1}{x}\,dx = C + \int \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} \cdots\right.$$

$$= C + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \frac{x^5}{5 \cdot 5!} +$$

$$= \boxed{C + \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} x^n, \quad R = \infty}$$