§11.9 #9 $\quad f(x) = \dfrac{1+x}{1-x}$ $\quad$ Use $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$

$|x| < 1$

Find a power series representation.
Find the radius of convergence.

$$\frac{1+x}{1-x} = (1+x)\left(\frac{1}{1-x}\right)$$

$$= (1+x)\sum_{n=0}^{\infty} x^n$$

$$= 1\cdot\sum_{n=0}^{\infty} x^n + x\sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$R = 1$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} \qquad -1 < x < 1$$

$$= (1 + x + x^2 + x^3 + \ldots) + (x + x^2 + x^3 + x^4 + \ldots)$$

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} 2x^n$$

$$= 1 + 2\sum_{n=1}^{\infty} x^n, \qquad R = 1$$

$-1 < x < 1$

§11.9 #17 $f(x) = \dfrac{x^3}{(x-2)^2}$

Find a power series and its radius of convergence.

$f(x) = x^3 \left(\dfrac{1}{(x-2)^2} \right)$

Use $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $R=1$

Differentiate.

$\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$, $R=1$

$\dfrac{1}{(x-1)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$, $R=1$

$\dfrac{4}{4(x-1)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$, $R=1$

$\dfrac{4}{(2(x-1))^2} = \dfrac{4}{(2x-2)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$ $R=1$

- $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)$
  $= \dfrac{d}{dx}(1-x)^{-1}$
  $= -1 \cdot (1-x)^{-2}(-1$
  $= \dfrac{1}{(1-x)^2}$
- Note $(a-b)^2 = (b-a)^2$ because $(a)^2 = (-a)^2$

Put in $\frac{x}{2}$ for x.

$\dfrac{4}{(2(\frac{x}{2})-2)^2} = \sum_{n=1}^{\infty} n \left(\dfrac{x}{2}\right)^{n-1}$, ~~R=1~~ $\left|\frac{x}{2}\right| < 1$
$|x| < 2$
$R=2$

$\dfrac{1}{(x-2)^2} = \dfrac{1}{4} \sum_{n=1}^{\infty} n \left(\dfrac{x}{2}\right)^{n-1}$,

$\dfrac{x^3}{(x-2)^2} = \dfrac{x^3}{4} \sum_{n=1}^{\infty} n \left(\dfrac{x}{2}\right)^{n-1}$

$\dfrac{x^3}{(x-2)^2} = \dfrac{1}{4} \sum_{n=1}^{\infty} n \dfrac{x^{n+2}}{2^{n-1}}$

$$= \frac{1}{2^2}\sum_{n=1}^{\infty} \frac{n\, x^{n+2}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{n\, x^{n+2}}{2^{n+1}}$$

$$= \frac{1\cdot x^3}{2^2} + \frac{2x^4}{2^3} + \frac{3x^5}{2^4} + \frac{4x^6}{2^5} + \cdots$$

$$\sum_{n=3}^{\infty} \frac{(n-2)x^n}{2^{n-1}}, \qquad R=2$$

§11.9 # 13

(a) Use differentiation to find a power series for $\frac{1}{(1+x)^2}$.

SOLUTION

Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n, \quad |-x|<1$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \quad |x|<1$$

Differentiate

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = -1\cdot(1+x)^{-2}$$

$$= \frac{-1}{(1+x)^2}$$

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$$

$$\frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$\frac{1}{(1+x)^2} = -1 \cdot \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+2} (n+1) x^n$$

$$\boxed{\begin{aligned}&(-1)^{n+2}\\ &= (-1)^n(-1)^2\\ &= (-1)^n\end{aligned}}$$

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

$R = 1$

b) Find a power series for $f(x) = \dfrac{1}{(1+x)^3}$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4$$

$= (1+x)^{-2}$ Differentiate

$$-2(1+x)^{-3} = \overset{n=0}{-2} + \overset{n=1}{3\cdot 2x} - \overset{n=2}{4\cdot 3x^2} + 5\cdot 4x^3 - 6\cdot 5x^4 + \cdots$$

$$\frac{-2}{(1+x)^3} = \sum_{n=0}^{\infty} (-1)^{n+1}(n+2)(n+1)x^n$$

OR

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)x^n, \quad R=1$$

differentiate

$$\frac{-2}{(1+x)^3} = \sum_{n=1}^{\infty} (-1)^n (n+1)n\, x^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1}(n+2)(n+1)x^n$$

Divide by -2

$$\frac{1}{(1+x)^3} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2} x^n, \quad R=1$$

(c) Find a power series for

$$\frac{x^2}{(1+x)^3}.$$

Multiply (b) by x^2

$$\frac{x^2}{(1+x)^3} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1) x^n}{2}, \quad R=1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1) x^{n+2}}{2}, \quad R=1$$

Re-index

$$\frac{x^2}{(1+x)^3} = \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1) x^n}{2}, \quad R=1$$

§11.10 Example: Find a power series for $\frac{1}{(1+x)^3}$; find R

Use

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-2)(k-3)}{3!}x^3 + \cdots$$

$$R = 1$$

$\cancel{k=}$

$$\frac{1}{(1+x)^3} = (1+x)^{-3}$$

$$k = -3$$

$$(1+x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} x^n = 1 + \frac{-3x}{1!} + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \cdots$$

$$R = 1$$

$$= \overset{n=0}{1} \; \overset{n=1}{- \frac{\cancel{1\cdot 2}\cdot 3}{\cancel{1\cdot 2}}x} \; \overset{n=2}{+ \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\; 2!}x^2} \; \overset{n=3}{- \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{1\cdot 2\; 3!}x^3}$$

$$\frac{1}{(1+x)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!}{2\, n!} x^n, \quad R = 1$$

§11.9 $\displaystyle\int \frac{x - \tan^{-1} x}{x^3}\,dx$

SOLUTION

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n, \qquad |-x^2| < 1$$

$$|x| < 1$$

$$R = 1$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

integrate

$$\tan^{-1} x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Put in $x=0$

$$0 = \tan^{-1}(0) = C + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1}}_{=0}$$

$$C = 0$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$$

$$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots \right)$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \frac{x^{11}}{11} \dots$$

$$\frac{x - \tan^{-1} x}{x^3} = \frac{1}{x^3}\left(\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \frac{x^{11}}{11} + \cdots \right.$$

$$= \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \frac{x^6}{9} + \frac{x^8}{11} - \frac{x^{10}}{13} + \cdots$$

$$\int \frac{x - \tan^{-1} x}{x^3}\, dx = C + \int \left(\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \frac{x^6}{9} + \frac{x^8}{11} - \frac{x^{10}}{13} + \cdots \right. d$$

$$= C + \underset{n=1}{\frac{x^1}{3\cdot 1}} - \underset{n=2}{\frac{x^3}{5\cdot 3}} + \underset{n=3}{\frac{x^5}{7\cdot 5}} - \underset{n=4}{\frac{x^7}{9\cdot 7}} + \frac{x^9}{11\cdot 9} - \frac{x^{11}}{13\cdot 11} + $$

$$= C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n-1)} x^{2n-1}$$

$R = 1$