§11.10 Continued

EXAMPLE Last time we found that

$$\int e^{-x^2}dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)}$$

(b) Evaluate $\int_0^1 e^{-x^2}dx$ correct to within an error of $0.001 = 10^{-3}$.

SOLUTION

$$\int_0^1 e^{-x^2}dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)}\right]_0^1$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots\right]_0^1$$

$$= \left(1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} + \cdots\right) - 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$$

This is an alternating series $b_n = \frac{1}{n!\,(2n+1)}$

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!(2n+1)} = 0$

(ii) $b_{n+1} \leq b_n$

$$\frac{1}{(n+1)!\,(2(n+1)+1)} \leq \frac{1}{n!\,(2n+1)}$$

$$\frac{n!\,(2n+1)}{(n+1)!\,(2n+3)} \le 1$$

$$\frac{1}{n+1}\left(\frac{(2n+1)}{(2n+3)}\right) \le 1$$

less than 1   less than 1

~~Remark~~ Remainder:

$$|R_n| \le b_{n+1} \le 10^{-3}$$

$$b_{n+1} = \frac{1}{(n+1)!\,(\underbrace{2(n+1)+1}_{2n+3})} \le \frac{1}{10^3}$$

$$b_n = \frac{1}{n!\,(2n+1)}$$

$$b_1 = \frac{1}{1!\,(2\cdot 1+1)} = \frac{1}{3}$$

$$b_2 = \frac{1}{2!\,(2\cdot 2+1)} = \frac{1}{10}$$

$$b_3 = \frac{1}{3!\,(2\cdot 3+1)} = \frac{1}{3\cdot 2\cdot 1\cdot 7} = 1$$

$$b_4 = \frac{1}{4!\,(2\cdot 4+1)} = \frac{1}{4\cdot 3\cdot 2\cdot 1\cdot 9} = .00462$$

$$b_{n+1} = b_5 = .0007 \checkmark$$

$$< 10^{-3}$$

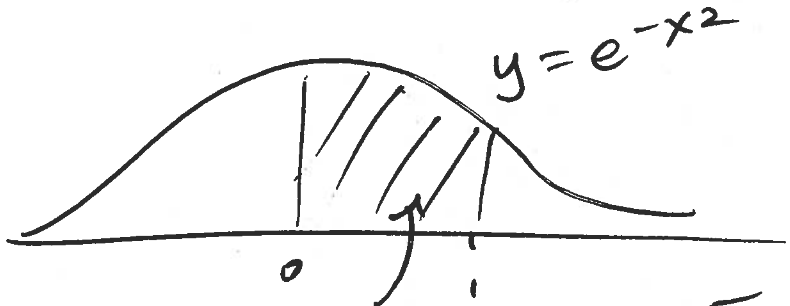
~~A =~~ $n+1 = 5$

$n = 4$

Add stop at $n=4$.

$$\sum_{n=0}^{4} \frac{(-1)^n}{n!(2n+1)}$$

$$= \frac{1}{0!(2\cdot 0+1)} - \frac{1}{1!(2\cdot 1+1)} + \frac{1}{2!(2\cdot 2+1)} - \frac{1}{3!(2\cdot 3+1)} + \frac{1}{4!(2\cdot 4+1)} \approx 0.7475$$



Area ≈ 0.7475
accurate to 10^{-3}

EXAMPLE Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Background

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$k=2 \qquad (1+x)^2 = 1 + 2x + x^2$

$k=3 \qquad (1+x)^3 = 1 + 3x + 3x^2 + x^3$

n is positive integer $(1+x)^n =$

$$= \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} x + \binom{n}{2} 1^{n-2} x^2 + \cdots \binom{n}{n} 1^0 x^n$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

SOLUTION

Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = (1+x)^k$$

$$f'(x) = k(1+x)^{k-1}$$

$$f''(x) = k(k-1)(1+x)^{k-2}$$

$$n=3 \quad f^{(3)}(x) = k(k-1)(k-2)(1+x)^{k-3}$$

$$\vdots$$

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$$

$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$$

So the Maclaurin Series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

Find the Radius of Convergence

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{k(k-1)(k-2)\cdots(k-n+1)(k-(n+1)+1)x^{n+1}}{(n+1)!}}{\left(\dfrac{k(k-1)\cdots(k-n+1)x^n}{n!}\right)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{k(k-1)(k-2)\cdots(k-n+1)} \right| \cdot \left| \frac{x^{n+1}}{x^n} \right| \cdot \left| \frac{n!}{(n+1)(n!)} \right|$$

$$= \lim_{n\to\infty} \underbrace{\left| \frac{k-n}{n+1} \right|}_{\to 1} |x| = |x| < 1$$

Radius of convergence $R = 1$.

<u>Notation</u> k choose n

$$\boxed{\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}}$$

e.g.

$k-n+1 = k-3+1 = k-2$

$$\binom{k}{3} = \frac{k(k-1)(k-2)}{3!}$$

five factors in numerator.

$$\binom{k}{5} = \frac{k(k-1)(k-2)(k-3)(k-4)}{5!}$$

We write

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \qquad |x| < 1}$$

EXAMPLE Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

SOLUTION

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{(4-x)^{1/2}} = \frac{1}{(4(1-\frac{x}{4}))^{1/2}}$$

$$= \frac{1}{4^{1/2}(1-\frac{x}{4})^{1/2}} = \frac{1}{2}\left(\frac{1}{(1+(-\frac{x}{4}))^{1/2}}\right)$$

$$= \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-1/2} \qquad \text{So } k = -\frac{1}{2}$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(-\frac{x}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{4^n} x^n$$

Let's simplify

$$\binom{-1/2}{n} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)\cdots(-\frac{1}{2}-n+1)}{n!}$$

$n=1$ $$\binom{-1/2}{1} = \frac{-1/2}{1!} = -\frac{1}{2}$$

$$\binom{-1/2}{2} = \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} = \frac{1\cdot 3}{2^2 \cdot 2!}$$

$$\binom{-\frac{1}{2}}{3} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} = \frac{-1\cdot 3\cdot 5}{2^3\cdot 3!}$$

$$\binom{-\frac{1}{2}}{4} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!} = \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}$$

$$\binom{-\frac{1}{2}}{n} = \frac{(-1)^n(1\cdot 3\cdot 5\cdot 7\cdots(2n-1))}{2^n\, n!}$$

$$\frac{1}{\sqrt{4-x}} = \sum_{n=0}^{\infty} \frac{(-1)^n(1\cdot 3\cdot 5\cdot 7\cdots(2n-1))}{2^n\, n!} \cdot \frac{(-1)^n}{4^n} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{8^n\, n!} x^n}$$

$$(-1)^n(-1)^n = (-1)^{2n}$$
$$= ((-1)^2)^n$$
$$= 1^n = 1$$

$$(-1)^n(-1)^n$$
$$(-1\cdot -1)^n = 1$$

Radius of Conv.

$$\left|\frac{-x}{4}\right| < 1$$

$$\frac{|x|}{4} < 1$$

$$|x| < 4$$

$$\boxed{R = 4}$$

Next week Test: §11.6 – 11.10