§11.10 Continued

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The partial sums are called Taylor Polynomials

$T_0(x) = f(a)$

$T_1(x) = f(a) + f'(a)(x-a)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

In general, $f(x)$ is the sum of its Taylor Series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

Let $R_n(x) = f(x) - T_n(x)$.

We have $f(x) = T_n(x) + R_n(x)$

$R_n(x)$ is called the remainder of the Taylor series.

<u>Theorem</u> If $f(x) = T_n(x) + R_n(x)$, and $\lim_{n \to \infty} R_n(x) = 0$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on $|x-a| < R$.

Overview

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}_{T_2(x)} + \underbrace{\frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}_{R_2(x)}$$

Taylor's Inequality If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a| \le d$.

Example: Prove that e^x is the sum of its Maclaurin series.

SOLUTION $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

For any $d > 0$, if $|x| \le d$,

then $|f^{(n+1)}(x)| = e^x \le e^d$.

By Taylor's Inequality with $a=0$, $M = e^d$, we get

$$\cancel{|R_n(x)|}\ |R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}, \quad |x| \le d$$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^d\,|x|^{n+1}}{(n+1)!} = 0$$

So $\lim_{n\to\infty} R_n(x) = 0$. So e^x is equal to the sum of its MacLaurin series,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- If we let $x=1$, we get

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

- Another formula for e:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Example: Find the Taylor series for $f(x)=e^x$ at $a=2$.

Taylor Series.

SOLUTION $$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)\,(x-a)^n}{n!}$$ $a=2$

$f(x)=e^x$

$f'(x)=e^x$

$f''(x)=e^x$

$f^{(n)}(x)=e^x$ $\qquad f^{(n)}(2)=e^2$

$$e^x=\sum_{n=0}^{\infty}\frac{e^2(x-2)^n}{n!}\qquad R=\infty$$

To show $R=\infty$, use the ratio Test.

Example: Find the Maclaurin series for $f(x) = \sin x$.

SOLUTION Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!}$$

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$	$f'(0) = \cos(0) = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$
$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -1$
$f^{(8)}(x) = \sin x$	$f^{(8)}(0) = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = f(0) + f'(0)\,x + \frac{f''(0)\,x^2}{2!} + \frac{f'''(0)}{3!}\,x^3$$
$$+ \frac{f^{(4)}(0)\,x^4}{4!} + \frac{f^{(5)}(0)\,x^5}{5!} + \frac{f^{(6)}(0)\,x^6}{6!}$$
$$+ \frac{f^{(7)}(0)\,x^7}{7!} + \cdots$$
$$= 0 + 1.x + 0 - \frac{1 \cdot x^3}{3!} + 0 + \frac{1 \cdot x^5}{5!} + 0 - \frac{1 \cdot x^7}{7!} + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Find the radius of convergence.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} |x^2| \cdot \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}$$

$$= \lim_{n\to\infty} \frac{|x^2|}{(2n+3)(2n+2)} = 0$$

$$\boxed{\begin{array}{c} R = \infty \\ (-\infty, \infty) \end{array}}$$

EXAMPLE Find the Maclaurin series for cos x.

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + -$$

$$R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

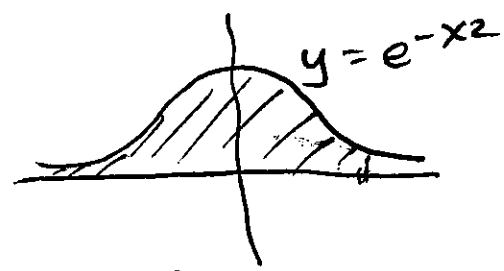
<u>EXAMPLE</u> Find the Maclaurin series for the function $f(x) = x\cos x$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$x\cos x = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

<u>Example</u>: Evaluate $\int e^{-x^2}dx$ as an infinite series.



$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)}{4!} \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + $$

$$R = \infty$$

$$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\, dx$$

$$= \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} \right. d$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(2n+1)} = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!}$$

$$- \frac{x^{11}}{11\cdot 5!} + \cdots$$

$$\int e^{-x^2}dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)}$$

§11.8 #6 $\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

$$\lim_{n\to\infty} \left| \frac{\sqrt{n+1}\, x^{n+1}}{\sqrt{n}\, x^n} \right|$$

$$= \lim_{n\to\infty} \underbrace{\sqrt{\frac{n+1}{n}}}_{\downarrow\infty} \left| \frac{x^{n+1}}{x^n} \right|$$

$$= |x| < 1 \qquad R = 1$$

$$-1 < x < 1$$

$\underline{x=1}$ $\sum_{n=1}^{\infty} \sqrt{n}$ divergent by Limit Comp Test

$x=-1$ $\sum_{n=1}^{\infty} \sqrt{n}(-1)^n$ div. by Limit Comp Test.

interval convergence
$(-1, 1)$

§11.8 # 8 $\sum_{n=1}^{\infty} n^n x^n$

Root Test

$$\lim_{n\to\infty} \sqrt[n]{|n^n x^n|} = \lim_{n\to\infty} n|x| = \infty > 1$$

$R = 0$

Interval convergent

$\{0\}$

§11.8 #9 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n}$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n^2 x^n}\right|$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^2}_{\to 1} |x| \cdot \frac{1}{2} = \frac{|x|}{2} < 1$$

$$\frac{|x|}{2} < 1$$

$$|x| < 2 \quad \boxed{R=2}$$

$$-2 < x < 2$$

endpoints

$x=-2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n^2 (-1)^n$$

$$= \sum_{n=1}^{\infty} n^2 \text{ div by test for div.}$$

$$\lim_{n\to\infty} n^2 = \infty$$

$x=2$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 2^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n^2$

$$\lim_{n\to\infty} (-1)^n n^2 \neq 0$$

Divergent by test for div.

interval of convergence

$$\boxed{(-2,2)}$$

§11.8 #19 $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

Root Test

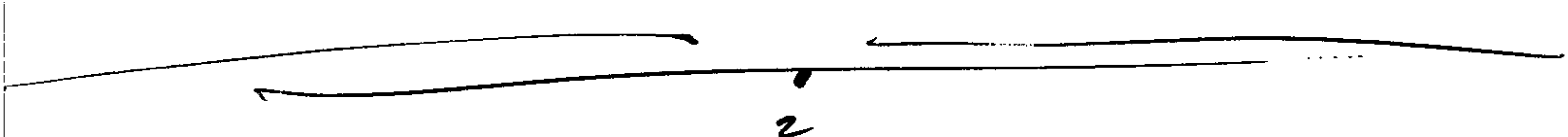
$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{(x-2)^n}{n^n} \right|}$$

$$= \lim_{n\to\infty} \frac{|x-2|}{n} = 0 < 1$$

$R = \infty$

interval convergence

$(-\infty, \infty)$.



§11.8 #23 $\sum_{n=1}^{\infty} n!\,(2x-1)^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n} \right|$$

$$= \lim_{n\to\infty} (n+1)\,|2x-1| = \infty > 1$$

$2x-1=0$
$2x=1$
$x=\frac{1}{2}$

$R=0$
interval conv.
$\{\frac{1}{2}\}$