§11.10 Taylor and MacLaurin Series

HW 11.10 # 5-20, 25-38, 47-50

Suppose that f is a function that can be represented as a power series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

$$|x-a| < R$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

<u>Note</u> $f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \dots$

$$= c_0$$

$$\boxed{c_0 = f(a)}$$

$$f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

put in $x=a$

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2$$

$$\boxed{c_1 = f'(a)}$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 +$$

put in $x=a$

$$f''(a) = 2c_2$$

$$\boxed{c_2 = \frac{f''(a)}{2}}$$

$$f'''(x) = 3\cdot 2\cdot 1\, C_3 + 4\cdot 3\cdot 2\, C_4 (x-a) + 5\cdot 4\cdot 3\, C_5 (x-a)^2 + \cdots$$

put in $x = a$

$$f'''(a) = 3\cdot 2\cdot 1\, C_3$$

$$C_3 = \frac{f'''(a)}{3\cdot 2\cdot 1} \quad \#$$

$$\boxed{C_3 = \frac{f^{(111)}(a)}{3!}}$$

$$\boxed{C_n = \frac{f^{(n)}(a)}{n!}}$$

when $n=0$, $0! = 1$, $f^{(0)} = f$

Theorem

If f has a power series representation $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ $\quad |x-a| < R$,

then $C_n = \dfrac{f^{(n)}(a)}{n!}$

we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

This is the Taylor Series of f about a.

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

When $a=0$, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \qquad |x| < R$$

This called a Maclaurin Series.

Example Find the MacLaurin series of the function $f(x)=e^x$ and its radius of convergence.

SOLUTION

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$f(0) = e^0 = 1$

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$f'''(0) = e^0 = 1$

$f^{(n)}(0) = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{\overbrace{f^{(n)}(0)}^{=1}}{n!} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Find the radius of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ratio test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

Abs. conv. for all x.

$R = \infty$.

You can now do §11.10 # 13–20

EXAMPLE Find a power series representation for $f(x) = \tan^{-1} x$

SOLUTION

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

Let's find a power series representation for $\frac{1}{1+x^2}$.

$$\frac{1}{1+x^2} = \qquad = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$|-x^2| < 1$$

$$x^2 < 1$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad |x| < 1$$

$$\tan^{-1} x = C + \int \frac{1}{1+x^2}\, dx$$

$$= C + \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$$

Solve for C.

$x=0$

$$\tan^{-1} 0 = C + 0 - \frac{0^3}{3} + \frac{0^5}{5} \cdots$$

$$C = \tan^{-1} 0 = 0$$

$$C = 0$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad R = 1$$

Extra

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Quiz Thurs 11.6, 11.7, 11.8

EXAMPLE

(a) Evaluate $\int \frac{1}{1+x^7}\,dx$ as a power series.

(b) Approximate $\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to within 10^{-7}

SOLUTION

a) $$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n, \quad |x^7|<1$$
$$|x|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

$$\int \frac{1}{1+x^7}\,dx = C + \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

b) $$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots\right]_0^{0.5}$$

$$= (.5) - \frac{(.5)^8}{8} + \frac{(.5)^{15}}{15} - \frac{(.5)^{22}}{22} + \cdots$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^8 \cdot \frac{1}{8} + \left(\frac{1}{2}\right)^{15} \frac{1}{15} - \left(\frac{1}{2}\right)^{22} \frac{1}{22} + \cdots$$

$$= \frac{1}{2} - \frac{1}{8}\cdot\frac{1}{2^8} + \frac{1}{15}\,\frac{1}{2^{15}} - \frac{1}{22}\,\frac{1}{2^{22}} + \cdots$$

$$+ \frac{(-1)^n}{(7n+1)\,2^{7n+1}} + \cdots$$

This is an alternating series

$$b_n = \frac{1}{(7n+1)\,2^{7n+1}}$$

$$|R_n| \le b_{n+1} < 10^{-7}$$

$$\frac{1}{(7(n+1)+1)\,2^{7(n+1)+1}} < 10^{-7}$$

$n=3$ $$\frac{1}{(7(3+1)+1)\,2^{7(3+1)+1}} = 6.4\times 10^{-11} < 10^{-7}$$

$n=3$

We use

$$\int_0^{.5} \frac{1}{1+x^7}\,dx \approx \underset{n=0}{\frac{1}{2}} - \underset{n=1}{\frac{1}{8\cdot 2^8}} + \underset{n=2}{\frac{1}{15\cdot 2^{15}}} - \underset{n=3}{\frac{1}{22\cdot 2^{22}}}$$

$$\approx 0.49951374$$

§11.8 # ~~23~~ 25 Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$$

SOLUTION

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x+1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(4x+1)^n} \right| \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} |4x+1| \underset{\substack{\downarrow \\ 1}}{\left(\frac{n}{n+1}\right)^2} = |4x+1|$$

$$|4x+1| < 1$$

$$4\left|x+\tfrac{1}{4}\right| < 1$$

$$\left|x+\tfrac{1}{4}\right| < \tfrac{1}{4}$$

Rad. Conv $R = \frac{1}{4}$

$$-\tfrac{1}{4} < x + \tfrac{1}{4} < \tfrac{1}{4}$$

$$-\tfrac{1}{2} < x < 0$$

Endpoints

$x = -\frac{1}{2}$ $\quad \sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(4(-\frac{1}{2})+1)^n}{n^2}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Alternating series.

$b_n = \frac{1}{n^2} > 0$

(i) $b_{n+1} < b_n$

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}$$

$$n^2 < (n+1)^2$$

$$n < n+1$$

$$0 < 1 \checkmark$$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0 \checkmark$

Convergent by Alt. series Test.

$x = 0$ $\quad \sum_{n=1}^{\infty} \frac{(4.0+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series
$p = 2$, conv.

Interval of Convergence $[-\frac{1}{2}, 0]$

$$-\frac{1}{2} \le x \le 0$$

Radius of Conv $R = \frac{1}{4}$.

§ 11.8 #18 $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

SOLUTION

$$\lim_{n\to\infty} \left| \frac{n+1}{4^{n+1}}(x+1)^{n+1} \cdot \frac{4^n}{n(x+1)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{4^n}{4^{n+1}} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right|$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)}_{\downarrow\ 1} \frac{1}{4} |x+1| = \frac{|x+1|}{4} < 1$$

$$\frac{|x+1|}{4} < 1$$

$$|x+1| < 4 \qquad R = 4$$

$$-4 < x+1 < 4$$

$$-5 < x < 3$$

Endpoints

$x=3$ $\quad \sum_{n=1}^{\infty} \frac{n}{4^n}(3+1)^n = \sum \frac{n\,4^n}{4^n} = \sum n$

divergent by Test for Diverg.

$x = -5 \quad \sum_{n=1}^{\infty} \frac{n}{4^n}(-5+1)^n = \sum_{n=1}^{\infty} \frac{n}{4^n}(-4)^n$

$= \sum_{n=1}^{\infty} n(-1)^n$

Div. by Test for Div

$\lim_{n\to\infty} (-1)^n n$ DNE

Interval Conv $-5 < x < 3$

$(-5, 3)$

Radius Conv $R = 4.$