§11.9 Representations of Functions as Power Series.

Homework 11.9 #3-18, 23-26

EXAMPLE $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$= \frac{1}{1-x}$ for $|x| < 1$

- Radius of convergence $R = 1$
- Interval of convergence $-1 < x < 1$ OR $(-1, 1)$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1}$$

EXAMPLE Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n,$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

$|x| < 1$

$|-x^2| < 1$

$x^2 < 1$

$|x| < 1$

$\boxed{-1 < x < 1}$

$$\boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1}$$

EXAMPLE Find a power series representation for $\frac{1}{x+2}$

SOLUTION

$$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2}\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}$$

$$\left|-\frac{x}{2}\right| < 1$$

$$\frac{|x|}{2} < 1$$

$$|x| < 2$$

$\boxed{\text{Int. Conv. } -2 < x < 2}$

$\boxed{\text{Radius of Conv. } R = 2}$

EXAMPLE Find a power series representation of $\frac{x^3}{x+2}$.

SOLUTION

$$\frac{x^3}{x+2} = x^3\left(\frac{1}{x+2}\right) = \frac{x^3}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{x^3}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right) = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \boxed{\frac{x^3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}},$$

$$\left|-\frac{x}{2}\right| < 1$$

$$\frac{|x|}{2} < 1$$

$$|x| < 2$$

$$\boxed{-2 < x < 2}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{2^{n+1}}}$$

$$= \frac{(-1)^0 x^3}{2^1} + \frac{(-1)^1 x^4}{2^2} + \frac{(-1)^2 x^5}{2^3} + \cdots$$

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty}\frac{(-1)^{n-1} x^n}{2^{n-2}}, \quad -2 < x < 2$$

$$R = 2$$

<u>Theorem</u> If the power series $\Sigma c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \dots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in equations (i) and (ii) are both R.

EXAMPLE Express $\frac{1}{(1-x)^2}$ as a power series by differentiating Equation 1. What is the radius of convergence?

SOLUTION

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n , \qquad |x|<1 \qquad R=1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \dots\right)$$

$$-1\cdot(1-x)^{-2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$-\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\frac{1}{(1-x)^2} = -\left(1 + 2x + 3x^2 + 4x^3 + \dots\right)$$

$$\frac{1}{(1-x)^2} = -\sum_{n=0}^{\infty}(n+1)x^n , \qquad R=1$$

EXAMPLE Find a power series representation for $\ln(1-x)$ and its radius of convergence.

SOLUTION

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad R=1$$

$|x|<1$

$-1<x<1$

$-1<-x<1$

$0<1-x<2$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\int \frac{1}{1-x}\,dx \overset{C+}{=} \int (1 + x + x^2 + x^3 + \dots)\,dx$$

$$-\ln|1-x| = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots, \quad R=1$$

~~$\ln(1-x) = C +$~~ Solve for C.

Let $x=0$

$$-\ln|1-0| = C + 0 + \frac{0^2}{2} \dots$$

$$-\ln 1 = C, \qquad C=0$$

$$-\ln(1-x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots, \quad R=1$$

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad R=1$$

$$\boxed{\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad R=1}$$

§11.7 #27 $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

Check for convergence or divergence.

$a_n = \frac{k \ln k}{(k+1)^3}$ $\qquad b_n = \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$

$k+1 \geq k$

$(k+1)^3 \geq k^3$

$\frac{1}{(k+1)^3} \leq \frac{1}{k^3}$

$0 \leq \frac{k \ln k}{(k+1)^3} \leq \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$

$\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x^2}\, dx$

$\int \frac{\ln x}{x^2}\, dx$ $\qquad \int u\,dv = uv - \int v\,du$

$u = \ln x$ $\qquad dv = x^{-2}dx$

$du = \frac{1}{x}dx$ $\qquad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$= -\frac{\ln x}{x} - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx = \frac{-\ln x}{x} + \int x^{-2}dx$

$= -\frac{\ln x}{x} + \frac{x^{-1}}{-1}$

$= -\frac{\ln x}{x} - \frac{1}{x} = \frac{-(\ln(x)+}{x}$

$$\lim_{t\to\infty}\left[-\frac{(1+\ln x)}{x}\right]_1^t = \overset{*}{\lim_{t\to\infty}\frac{-(1+\ln t)}{t}} - \left(-\frac{1+\ln 1}{1}\right)$$

$$* \quad \lim_{t\to\infty} -\frac{(1+\overset{\nearrow\infty}{\ln t})}{\underset{\to\infty}{t}} \overset{LH}{=} \lim_{t\to\infty} -\frac{(0+\overset{\nearrow 0}{1/t})}{1} = \frac{0}{1} = 0$$

$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1$ So convergent by int test.

Therefore $\sum_{k=1}^{\infty} \frac{k\ln k}{(k+1)^3}$ is conv. by

comparison Test.