§11.8 Power Series

HW §11.8 # 3-28

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

In general, a power series centered at a is of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

EXAMPLE $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ convergent for $|x|<1$

$= \frac{1}{1-x}$

Geom series $a=1$, $r=x$

$S = \frac{a}{1-r}$

$|r|<1$

EXAMPLE For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

$= 1 + 1!\,x + 2!\,x^2 + 3!\,x^3 + \cdots$

SOLUTION

Ratio Test $\quad a_n = n!\, x^n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} \left| \frac{\cancel{x^n}\cdot x}{\cancel{x^n}} \right|$$

Aside
$6! = 6(5\cdot 4\cdot 3\cdot 2\cdot 1)$
$= 6\cdot 5!$

$$= \lim_{n\to\infty} (n+1)|x| = \infty > 1$$

for $x \neq 0$

Divergent for $x \neq 0$
convergent if $x = 0$.

- Radius of convergence $R = 0$.
- Interval of convergen[ce] $\{0\}$

EXAMPLE For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$ convergent?

SOLUTION

Ratio Test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-3)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} |x-3| \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}}$

$= \lim_{n\to\infty} \frac{|x-3|}{\underbrace{n+1}_{\to\infty}} = 0 < 1$

$L = 0 < 1$

Abs. Conv. for all x.

- Radius of convergence $R = \infty$
- Interval of convergence $(-\infty, \infty)$

Example For what values of x does the series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converge?

SOLUTION

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\right|\frac{n}{n+1} = |x-3|$$

$$\frac{n}{n+1} \to 1$$

Abs Conv when $|x-3| < 1$ ← This is called the radius of convergence

Div. when $|x-3| > 1$

when $|x-3| = 1$, the Ratio test is inconclusive.

$$|x-3| = 1$$
$$x-3 = 1 \quad \text{or} \quad x-3 = -1$$
$$x = 4 \quad \text{or} \quad x = 2$$

check

- $x = 2$

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

Alt. Series Test.

$b_n = \frac{1}{n}$

- $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$ ✓
- $b_{n+1} \le b_n$

  $\frac{1}{n+1} \le \frac{1}{n}$

  $n \le n+1$

  $0 \le 1$ ✓

Convergent by A.S.T.

- $x = 4$

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(1)^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$$

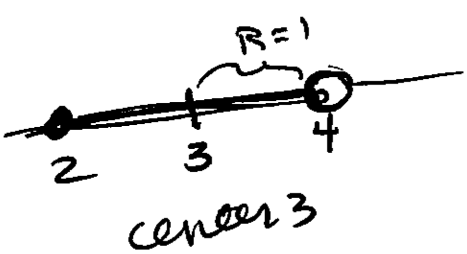
DIV. harmonic ($p=1$)

So $\overset{\text{abs}}{\text{convergent}}$ for $|x-3| < 1 \leftarrow R=1$

i.e. $-1 < x-3 < 1$

$2 \le x < 4$

Answer:
<u>Convergent</u> for $2 \le x < 4$



- The interval of convergence is $[2, 4)$
- The radius of conv. is $R=1$

Theorem For a given power series
$\sum_{n=0}^{\infty} c_n (x-a)^n$, then there are
only three possibities.

(i) The series converges when $x=a$.

(ii) The series converges for all
$x=a$.

(iii) There is a positive number R
such that the series
converges if $|x-a| < R$
and diverges if $|x-a| > R$.

§ 11.6 #6 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ check for abs. conv., condit. conv., or divergence.

- The Ratio Test is inconclusive for Rational Functions.

  check it out:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)^4}\cdot\frac{n^4}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n^4}{(n+1)^4} = \lim_{n\to\infty}\Big(\underbrace{\frac{n}{n+1}}_{\to 1}\Big)^4 = 1^4 =$$

inconclusive.

- Check for Abs. Convergence directly

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^4}\right| = \sum_{n=1}^{\infty}\frac{1}{n^4}$$ p-series $p = 4 > 1$ Convergent.

So $\sum \frac{(-1)^n}{n^4}$ is abs conv.

§11.6 #10 $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

- This is like a rational function. The Ratio Test will be inclusive.
- Check for Absolute Convergence directly.

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{\sqrt{n^3+2}} \right| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$$

$$a_n = \frac{n}{\sqrt{n^3+2}}, \quad b_n = \frac{n}{\sqrt{n^3}} = \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ div. $p = \frac{1}{2} < 1$ p-series

Limit Comparison Test

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\cancel{n}}{\sqrt{n^3+2}} \cdot \frac{\sqrt{n^3}}{\cancel{n}}$$

$$= \lim_{n \to \infty} \sqrt{\underbrace{\left(\frac{n^3}{n^3+2}\right)}_{\to 1}} = \sqrt{1} = 1$$

So $\sum \frac{n}{\sqrt{n^3+2}}$ is ~~conv~~ div.

So $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+2}}$ is <u>not</u> abs. conv

- Check for condition convergence.

$$\sum \frac{(-1)^n n}{\sqrt{n^3+2}}$$

Alt. Series Test

$$b_n = \frac{n}{\sqrt{n^3+2}}$$

- $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = 0$

- $b_{n+1} \le b_n$

$$\frac{n+1}{\sqrt{(n+1)^3+2}} \le \frac{n}{\sqrt{n^3+2}}$$

$$\cancel{\frac{\sqrt{n^3+2}}{\sqrt{(n+1)^3+2}} \le \frac{n}{n+1}}$$

$$(n+1)\sqrt{n^3+2} \le n\sqrt{(n+1)^3+2}$$

dead end.

$$f(x) = \frac{x}{\sqrt{x^3+2}} = \frac{x}{(x^3+2)^{1/2}}$$

$$f'(x) = \frac{1\cdot\sqrt{x^3+2} - x\,\frac{1}{2}(x^3+2)^{-1/2}\cdot 3x^2}{(\sqrt{x^3+2})^2}$$

$$= \frac{\sqrt{x^3+2} - \dfrac{\frac{3}{2}x^3}{\sqrt{x^3+2}}}{x^3+2} \qquad x \geq 0$$

x^3+2 ← pos $x \geq 0$

§11.6 #10 cont

$$\sqrt{x^3+2} - \frac{3}{2}\frac{x^3}{\sqrt{x^3+2}} \overset{?}{<} 0$$

$$\sqrt{x^3+2} \overset{?}{<} \frac{3}{2}\frac{x^3}{\sqrt{x^3+2}}$$

$$\sqrt{x^3+2}\,\sqrt{x^3+2} \overset{?}{<} \frac{3}{2}x^3$$

$$x^3+2 \overset{?}{<} \frac{3}{2}x^3$$

$$2 < \frac{1}{2}x^3$$

$4 < x^3$ yes for $x \geq 2$

Decreasing.

$\therefore \sum \frac{(-1)^n n}{\sqrt{n^3+2}}$ conv. by A.S.T

so Conditionally Convergent.