§11.6 Continued

Proof of the Ratio Test

Part 1

Assume $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$

we want to show that $\sum a_n$ is absolutely convergent.

Let r be a number such that

$$L < r < 1$$

then there exist $N > 0$ such that if $n \geq N$, then

$$\left| \frac{a_{n+1}}{a_n} \right| < r$$

i.e. $\frac{|a_{n+1}|}{|a_n|} < r$

i.e. $|a_{n+1}| < r|a_n|$ for $n \geq N$

So $|a_{N+1}| < r\,|a_N|$

$|a_{N+2}| < r\,|a_{N+1}| < r \cdot r|a_N| = r^2|a_N|$

$|a_{N+3}| < r|a_{N+2}| < r \cdot r^2|a_N| = r^3|a_N|$

we get

$|a_{N+k}| < r^k|a_N|$

$\sum_{k=1}^{\infty} r^k |a_N|$ is a geom series. $|r|<1$, so it is convergent.

by the Comp Test $\sum_{k=1}^{\infty} |a_{N+k}|$ is therefore convergent.

We have

$$\sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{N} |a_n|}_{\text{finite}} + \sum_{n=N+1}^{\infty} |a_n|$$

$$= \underbrace{\sum_{n=1}^{N} |a_n|}_{\text{finite}} + \underbrace{\sum_{k=1}^{\infty} |a_{N+k}|}_{\text{convergent by above.}}$$

So $\sum |a_n|$ is convergent.

So $\sum a_n$ is absolutely convergent.

§11.7 Strategies for Testing Series

Hw §11.7 #1-38

Strategies

1. If the the series is of the form $\sum \frac{1}{n^p}$ then it is a p-series. Convergen $p > 1$, divergent otherwise.

2. If the series is of the form $\sum ar^{n-1}$ then it is a Geometric Series, convergent if $|r| < 1$, divergent otherwise. o/w

3. If it is a positive series, then try using the Comparison Test or the Limit Comparison Test, and compare to a p-series or geom. series.

4. If $\lim_{n \to \infty} a_n \neq 0$, then the series diverges by the Limit Comparison Test.

(5) If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then use the Alternating Series Test.

(6) For Series that involve factorials or products, constants raised to the n power, use the Ratio Test.

<u>Note</u> For p-series $\sum \frac{1}{n^p}$, the ratio test gives $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$

So the ratio test is inconclusive. The the same is true for series involving ration functions of n like $\sum \frac{n+1}{n^3+2n+3}$. The ratio test will be inconclusive. (Try the comparison Test).

(7) <u>If</u> a_n is of the form $(b_n)^n$, then the Root Test may be used.

(8) <u>If</u> $a_n = f(n)$ where $\int_1^\infty f(x)dx$ is <u>easily</u> evaluated, then use the integral.

EXAMPLE $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Comp Test

$a_n = \frac{1}{2+3^n}$ $\quad b_n = \frac{1}{3^n}$

$0 \le a_n \le b_n$

$\frac{1}{2+3^n} \overset{?}{\le} \frac{1}{3^n}$

$2+3^n \geq 3^n$

$0 \le \frac{1}{2+3^n} \le \frac{1}{3^n}$

$\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$

$r = \frac{1}{3}$

$\left|\frac{1}{3}\right| < 1$

Conv. Geom Series.

by Comp. Test Convergent.

EXAMPLE $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Ratio Test

$\lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$

$= \lim_{n\to\infty} \left| \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} \right|$

$= \lim_{n\to\infty} \frac{2^{k+1}}{2^k} \frac{k!}{(k+1)!}$

$= \lim_{n\to\infty} 2 \cdot \left(\frac{1}{k+1} \right) = 0 < 1$

Conv. by Ratio Test.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

SOLUTION Alternating Series Test.

$$b_n = \frac{n^3}{n^4+1}$$

(i) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^3}{n^4+1} = 0$

(ii) $b_{n+1} \le b_n$

$$\frac{(n+1)^3}{(n+1)^4+1} \le \frac{n^3}{n^4+1}$$

$$(n^4+1)(n+1)^3 \le n^3\left((n+1)^4+1\right)$$

$$f(x) = \frac{x^3}{x^4+1}$$

$$f'(x) = \frac{3x^2(x^4+1) - x^3(4x^3)}{(x^4+1)^2}$$

$$= \frac{3x^6 + 3x^2 - 4x^6}{(x^4+1)^2}$$

$$= \frac{-x^6 + 3x^2}{(x^4+1)^2}$$

$$= \frac{\overset{\text{pos}}{x^2}\,(\overset{\text{neg}}{-x^4+3})}{\underset{\text{pos}}{(x^4+1)^2}} < 0$$

f is decreasing.

$\therefore$ Convergent by Alt. Series Test.

Example $\sum_{n=1}^{\infty} n e^{-n^2} = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{e^{n^2}}{e^{(n+1)^2}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{e^{n^2}}{e^{n^2+2n+1}}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)}_{\to 1} \underbrace{\frac{1}{e^{2n+1}}}_{\to 0} = 0 < 1$$

$L = 0 < 1$ Abs conv.

Try the integral test

$$\int_1^{\infty} x e^{-x^2} dx$$

$$= \lim_{t\to\infty} \int_1^t x e^{-x^2} dx$$

$u = -x^2$
$du = -2x\,dx$
$-\frac{1}{2} du = x\,dx$

$-\frac{1}{2} \int e^u du = -\frac{1}{2} e^u$
$= -\frac{1}{2} e^{-x^2}$

$$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \underbrace{-\frac{1}{2} e^{-t^2}}_{\to 0} - \left(-\frac{1}{2} e^{-1^2}\right) = \frac{1}{2e}$$

Convergent.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

$$\lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \quad \therefore \text{ divergent by Test for Divergence.}$$

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

$$a_n = \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \qquad b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

(get rid of lower degree terms.)

$$\sum b_n = \frac{1}{3}\sum \frac{1}{n^{3/2}} \quad p = 3/2 > 1$$

convergent

Limit Comp Test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \cdot \frac{3n^3}{\sqrt{n^3}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^3+1}}{\sqrt{n^3}} \cdot \frac{3n^3}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} \underbrace{\sqrt{\frac{n^3+1}{n^3}}}_{\to 1} \cdot \underbrace{\frac{3n^3}{3n^3+4n^2+2}}_{\to \frac{3}{3} = 1} = 1 \quad \text{(not 0, not } \infty)$$

$\therefore \sum a_n$ is ~~divergent~~ ~~conv.~~ convergent by Limit Comp Test.

§11.6 #7 Determine whether the series is absolutely convergent, conditionally convergent, or divergent

$$\sum_{K=1}^{\infty} K\left(\frac{2}{3}\right)^K$$

<u>SOLUTION</u> Ratio Test.

$$a_K = K\left(\frac{2}{3}\right)^K$$

$$\lim_{K\to\infty}\left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty}\frac{(K+1)\left(\frac{2}{3}\right)^{K+1}}{K\left(2/3\right)^K}$$

$$= \lim_{K\to\infty}\left(\frac{K+1}{K}\right)\left(\frac{2}{3}\right) = 1\cdot\frac{2}{3} = \frac{2}{3} < 1$$

$L = 2/3 < 1$ Abs. Conv. by Ratio Test.