§11.6 Absolute Convergence and the Ratio and Root Test.

HW §11.6 #1-28

Test §11.1-11.5

High 100

90-100	7	28%
80-89	4	16%
70-79	4	16%
60-69	6	24%
0-59	4	16%
	25	

Definition:

A series Σa_n is called <u>absolutely convergent</u> if the series of absolute values $\Sigma |a_n|$ is convergent.

EXAMPLE The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

← this series is convergent (p-series, $p=2$)

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

This series is convergent because by Alt. Series Test:

$b_n = \frac{1}{n}$

(i) $b_{n+1} \le b_n$

$\frac{1}{n+1} \le \frac{1}{n}$

$n \le n+1$

$0 \le 1$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So convergent by Alt. Series Test.

However, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

is <u>not</u> absolutely convergent because

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ which is the harmonic series, therefore divergent.

<u>Definition</u> : A series $\sum a_n$ is called <u>conditionally</u> <u>convergent</u> if it is convergant but not absolutely convergent.

The previous example, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, is conditionally convergent.

<u>Theorem</u> If a series $\sum a_n$ is absolutely convergent, then it is convergent.

<u>Proof</u> Suppose $\sum a_n$ is absolutely convergent.

$$a_n \le |a_n|$$

$$0 \le a_n + |a_n| \le 2|a_n|$$

~~Thes~~ The series $\sum 2|a_n| = 2\sum |a_n|$ is convergent because we are assuming that $\sum a_n$ is absolutely convergent.

By the comparison test $\sum (a_n + |a_n|)$ is therefore convergent.

$$\sum a_n = \sum(a_n + |a_n| - |a_n|)$$

$$= \sum(a_n + |a_n|) - \sum |a_n|$$

both series are convergent so $\sum a_n$ is convergent. $\square$

EXAMPLE Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

SOLUTION Let's check for absolute convergent.

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

The comparison test gives us

$$0 \leq |\cos n| \leq 1$$

- $$0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

- $\sum \frac{1}{n^2}$ is convergent, p-series, $p = 2$.

$\therefore$ $\sum \left| \frac{\cos n}{n^2} \right|$ is convergent by the comparison test.

Therefore $\sum \frac{\cos n}{n^2}$ is absolutely convergent, and the therefore convergent.

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n.

EXAMPLE: Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\left((-1)^{n+1} \frac{(n+1)^3}{3^{n+1}} \right)}{\left(\frac{(-1)^n n^3}{3^n} \right)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n} \right)^3}_{\to 1} \frac{1}{3} = \frac{1}{3} < 1$$

$L = \frac{1}{3} < 1$ Therefore abs. convergent, therefore convergent.

EXAMPLE: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

SOLUTION Let's use the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}$$

Aside

$(n+1)! = (n+1)\cdot n \cdot (n-1) \cdots 3 \cdot 2 \cdot$

$= (n+1)\, n!$

so

$$\frac{n!}{(n+1)!} = \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cancel{(n+1)}}{n^n} \cdot \frac{1}{\cancel{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n}{n} + \frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\boxed{\ln\left(1+\frac{1}{x}\right)^x}}$$

1^∞ indet. form

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^{x}$$

$$= \lim_{x\to\infty} \underset{\to\infty}{x} \cdot \underset{\to \ln 1 = 0}{\ln\left(1+\frac{1}{x}\right)}$$

$\infty \cdot 0$ form

$$= \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \quad \begin{matrix}\to \ln 1 = 0 \\ \to 0\end{matrix}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1+1/x}\right)\left(-\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$$

$$= \lim_{x\to\infty} \frac{1}{1+1/x} = 1$$

$\to 0$

Answer $e^1 = e$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

$L = e > 1$

So the series is divergent. □

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

EXAMPLE Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

SOLUTION We use the Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|}$$

$$= \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

$L = \frac{2}{3} < 1$ So the series is abs. convergent by the Root Test, and therefore convergent.