§11.2 #17 $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

Method I Geom Series $\displaystyle\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$

$|r|<1$

$$= \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^{n-1}\cdot 4}$$

$$= \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^{n-1}$$

$a = \frac{1}{4}$, $r = \frac{-3}{4}$

$$S = \frac{a}{1-r} = a\left(\frac{1}{1-r}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1-\frac{-3}{4}}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1+\frac{3}{4}}\right)\frac{4}{4}$$

$$= \frac{1}{4}\left(\frac{4}{4+3}\right) = \frac{1}{7}$$

Method II Expand

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{(-3)^{1-1}}{4^1} + \frac{(-3)^{2-1}}{4^2} + \frac{(-3)^{3-1}}{4^3} + \cdots$$

$$= \frac{1}{4} - \frac{3}{4^2} + \frac{3^2}{4^3} - \cdots$$

$a = \frac{1}{4}$, $r = \frac{a_2}{a_1} = \frac{-3}{4}$

$$s = a\left(\frac{1}{1-r}\right) = \frac{1}{7}$$

§11.2 # 29 $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

$$\lim_{n\to\infty} \ln\left(\underbrace{\frac{n^2+1}{2n^2+1}}_{\to 1/2}\right) = \ln\left(\tfrac{1}{2}\right) \neq 0$$

Divergent by Test for Divergence.

State the Test for Divergence:

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE, then $\sum_{n=1}^{\infty} a_n$ is divergent.

§11.5 #9 $\sum_{n=1}^{\infty} (-1)^n \frac{n}{10^n}$

$a_n = (-1)^n \frac{n}{10^n}$

$b_n = |a_n| = \left|\frac{(-1)^n n}{10^n}\right|$

$= \frac{n}{10^n}$

Show Convergent.

(o) $b_n = \frac{n}{10^n} > 0$

(i) $b_{n+1} \le b_n$

$\frac{n+1}{10^{n+1}} \le \frac{n}{10^n}$

$(n+1)10^n \le n\,10^{n+1}$

$n10^n + 10^n \le n10^n \cdot 10^1$

$10^n \le n10^n \cdot 10 - n10^n$

$10^n \le n10^n(10-1)$

$10^n \le n10^n(10-1)$

$1 \le n \cdot 9$

$1 \le 9n$

$\frac{1}{9} \le n$ ✓

OR

$f(x) = \frac{x}{10^x}$. $f(n) = b_n$ for all $n \geq 1$

$f'(x) = \frac{(1)10^x - x10^x \ln 10}{(10^x)^2}$

$$= \frac{10^x(1 - x\ln 10)}{10^{2x}}$$

$$= \frac{1 - x\ln 10}{10^x}$$

$$1 - x\ln 10 < 0$$

$$1 < x\ln 10$$

$$\frac{1}{\ln 10} < x \quad \text{true for } x \geq 1$$

so $f'(x) < 0$ for $x \geq 1$.

~~So f~~, So f is decreasing,

$\Rightarrow$ b_n is decreasing.

(ii) $\displaystyle \lim_{n\to\infty} \frac{n}{10^n} = \lim_{x\to\infty} \frac{x}{10^x} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{10^x \ln 10}$

$= 0$ ✓

$\therefore$ Convergent by Alternating Series Test.

Prove $b_n = \frac{n}{10^n}$ is decreasing using proof by induction.

Prove $b_{n+1} \le b_n$

~~base case~~

Base Case $n=1$

$$b_{1+1} \le b_1$$
$$b_2 \le b_1$$
$$\frac{2}{10^2} \le \frac{1}{10}$$
$$\frac{2}{100} \le \frac{1}{10}$$
$$\frac{1}{50} \le \frac{1}{10} \checkmark$$

Assume true for $n=k$. Show true for $n=k+1$.

Assume $b_{k+1} \le b_k$

$$\frac{k+1}{10^{k+1}} \le \frac{k}{10^k}$$
$$\frac{k+1}{10^{k+1}} \cdot \frac{1}{10} \le \frac{k}{10^k} \cdot \frac{1}{10}$$
$$\frac{k+1}{10^{k+2}} \le \frac{k}{10^{k+1}}$$
$$\frac{k+1}{10^{k+2}} + \frac{1}{10^{k+2}} \le \frac{k}{10^{k+1}} + \frac{1}{10^{k+2}}$$

$$\frac{k+2}{10^{k+2}} \le \frac{k}{10^{k+1}} + \frac{1}{10^{k+1}} \cdot \frac{1}{10} \le \frac{k}{10^{k+1}} + \frac{1}{10^{k+1}}$$

$$\frac{k+2}{10^{k+2}} \le \frac{k}{10^{k+1}} + \frac{1}{10^{k+1}}$$

$$\frac{k+2}{10^{k+2}} \le \frac{k+1}{10^{k+1}}$$

$$\frac{(k+1)+1}{10^{(k+1)+1}} \le \frac{k+1}{10^{k+1}}$$

$$b(k+1)+1 \le b(k+1) \checkmark$$

§11.5 $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$ $(|\text{error}| < 5 \times 10^{-6})$

a) Show Convergent

(a) $b_n = \frac{1}{10^n n!}$

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{10^n n!} = 0$

$\downarrow \infty \quad \downarrow \infty$

(ii) $b_{n+1} \le b_n$

$$\frac{1}{10^{n+1}(n+1)!} \le \frac{1}{10^n n!}$$

$$10^n n! \le 10^{n+1}(n+1)!$$

$$10^n n! \le 10^n \cdot 10\,(n+1)\,n!$$

$$1 \le 10(n+1)$$

$$\frac{1}{10} \le n+1$$

$$\frac{1}{10} - 1 \le n \quad \checkmark$$

$\therefore$ Convergent by Alt. Series Test.

Find how large n should by so that $|R_n| < 5 \times 10^{-6}$

$$|R_n| \le b_{n+1} \le 5 \times 10^{-6}$$

$$\frac{1}{10^{n+1}(n+1)!} = \frac{5}{10^6}$$

$$10^6 \le 5 \cdot 10^{n+1}(n+1)!$$

$$\frac{10^6}{5} \le 10^{n+1}(n+1)!$$

$$\frac{10^5 \cdot \cancel{10}}{5} \le 10^n \cdot \cancel{10}(n+1)!$$

$$\frac{10^5}{5} \le 10^n(n+1)!$$

$$\frac{100{,}000}{5} \le 10^n(n+1)!$$

$$20{,}000 \le 10^n(n+1)!$$

n	$10^n(n+1)!$
1	$10^1(1+1)!$
2	
3	$10^3(3+1)! = (1000)4\cdot3\cdot2\cdot1 = 24{,}000$ ✓
4	$10^4(4+1)! = (10{,}000)5\cdot4\cdot3\cdot2\cdot1 = 120{,}000$
5	$10^5(5+1)! =$

$\boxed{n=3}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$$

$$= \frac{(-1)^0}{10^0 0!} + \frac{(-1)^1}{10 \cdot 1!} + \frac{(-1)^2}{10^2 \cdot 2!} + \frac{(-1)^3}{10^3 3!} + \frac{(-1)^4}{10^4 4!}$$

$$+ \frac{(-1)^5}{10^5 5!}$$

$$= \underset{b_0}{1} \underset{-b_1}{- \frac{1}{10}} \underset{+b_2}{+ \frac{1}{200}} \underset{-b_3}{- \frac{1}{6000}} \underset{+b_3}{+ \frac{1}{240{,}000}}$$

$$- \frac{1}{}$$

$$= 4.2 \times 10^{-6}$$

$n=3$

$$b_3 \leq 5 \times 10^{-6}$$

$n=3$

11.2 1a) What is the difference between a sequence and a series?

A sequence is an ordered list of numbers. A series is the sum of an ordered list of numbers.

§11.2 #35 Determine whether the telescoping series converges. Find its sum if it is convergent.

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

$$\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$$

$$2 = A(n+1) + B(n-1)$$

$n = 1$

$2 = 2A, \quad A = 1$

$n = -1$

$2 = -2B, \quad B = -1$

$$= \frac{1}{n-1} - \frac{1}{n+1}$$

$$\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$S_2 = \frac{1}{2-1} - \frac{1}{2+1} = 1 - \frac{1}{3}$$

$$\cancel{S_3 = \frac{1}{3-1} - \frac{1}{3+1} = \frac{1}{2} - \frac{1}{4}}$$

$$\cancel{S_3 = S_2 +} \quad S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$S_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$S_5 = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{5}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{6}\right)$$

$$= 1 + \frac{1}{2} - \cancel{\frac{1}{6}}\,\frac{1}{5} - \frac{1}{6}$$

$$S_6 = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{6}\right) + \left(\cancel{\tfrac{1}{5}} - \tfrac{1}{7}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} - \overset{\to 0}{\frac{1}{n}} - \overset{\to 0}{\frac{1}{n+1}}\right) = 1 + \frac{1}{2} = \frac{3}{2}.$$