Test Thurs. 11.1 - 11.5

§ 11.5 #23 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}, \quad (|\text{error}| < 0.00005 = 5 \times 10^{-5})$$

SOLUTION Alternating Series Test.

(0) $b_n = \frac{1}{n^6} > 0$

(i) $b_{n+1} \leq b_n$

$$\frac{1}{(n+1)^6} \leq \frac{1}{n^6}$$

$$n^6 \leq (n+1)^6$$

$$n \leq n+1$$

$$0 \leq 1 \quad ✓$$

(ii) $\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty} \frac{1}{n^6} = 0$ ✓

Therefore convergent by the Alternating Series Test. (A.S.T)

The Alt. Series Estimation Theorem.

$|R_n| \le b_{n+1}$

$|R_n| \le b_{n+1} \le 5 \times 10^{-5}$

Find n.

$\frac{1}{(n+1)^6} \le 5 \times 10^{-5}$

$\frac{1}{5 \times 10^{-5}} \le (n+1)^6$

$\frac{1}{5} \times 10^5 \le (n+1)^6$

$.2 \times 10^5 \le (n+1)^6$

$2 \times 10^4 \le (n+1)^6$

$(2 \times 10^4)^{1/6} \le n+1$

$(2 \times 10^4)^{1/6} - 1 \le n$

$4.21 \le n$

$\boxed{5 \le n}$

§11.3 #19 $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Determine whether the series is convergent or divergent.

The Comparison Test or Limit Comparison Test is the easiest way to answer this question. However, because this is the section on the integral test, let's use the integral.

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3}\,dx$$

$\int \frac{\ln x}{x^3}\,dx$ integrate by parts

$u = \ln x$ $\qquad dv = x^{-3}\,dx$

$du = \frac{1}{x}\,dx$ $\qquad v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

$$\int u\,dv = uv - \int v\,du$$
$$= (\ln x)\left(\frac{-1}{2x^2}\right) - \int \left(-\frac{1}{2x^2}\right)\frac{1}{x}\,dx$$
$$= \ln x\left(\frac{-1}{2x^2}\right) + \frac{1}{2}\int x^{-3}\,dx$$
$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\,\frac{x^{-2}}{-2} + C$$
$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^t = \lim_{t\to\infty}\Big[\underbrace{-\frac{\ln t}{2t^2}}_{\mathrm{I}} - \underbrace{\frac{1}{4t^2}}_{\mathrm{II}}\Big] - \left[-\frac{\overbrace{\ln 1}^{=0}}{2(1)^2} - \frac{1}{4(1)^2}\right]$$

(I) $$\lim_{t\to\infty} \frac{-\ln t}{2t^2} \overset{LH}{=} \lim_{t\to\infty} \frac{-1/t}{4t} = \lim_{t\to\infty} \frac{-1}{t}\cdot\frac{1}{4t}$$

$$= \lim_{t\to\infty} -\frac{1}{4t^2} = 0$$

(I) $$\lim_{t\to\infty} -\frac{1}{4t^2} = 0$$

Convergent by the Integral Test.

§ 11.3 Is it convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

Use the integral test.

$$\int_1^{\infty} \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{e^{1/t}}{t^2}\,dt$$

$$\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u = -e^{1/x} + C$$

subst. method

$u = \frac{1}{x} = x^{-1}$

$du = -1 \cdot x^{-2}\,dx$

$du = -\frac{1}{x^2}\,dx$

$-du = \frac{1}{x^2}\,dx$

$$= \lim_{t\to\infty} \left[-e^{\frac{1}{x}}\right]_1^t = \lim_{t\to\infty} -e^{1/t} - -e^{1/1}$$

$$= -e^0 + e = -1 + e$$

Convergent by Integral Test.

State the Test for Divergence.

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \ ? \quad \frac{a}{1-r}$$

For which values of r is it convergent?

$|r| < 1$

i.e.

$-1 < r < 1$

Example: Find the sum.

$$\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^1\left(\frac{2}{3}\right)^{n-1}$$

$$= 2\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$$

$a = 2, \quad r = \frac{2}{3}$

$$s = \frac{a}{1-r} = \frac{2}{1-2/3} = \frac{2}{(1/3)} = 2\left(\frac{3}{1}\right)$$

$$= 6$$

EXAMPLE: Is the series convergent?

$$\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+n+5}$$

SOLUTION $\lim_{n\to\infty} \frac{n^2+1}{3n^2+n+5} = \frac{1}{3}.$

The series Diverges by the Test for Divergence.

§11.4 Use the comparison test or the limit comparison test to determine whether the series is convergent.

#12 $\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$

$a_n = \frac{1+\sin n}{10^n}$

$b_n = \frac{2}{10^n}$

$0 \le \frac{1+\sin n}{10^n} \le \frac{1+1}{10^n} = \frac{2}{10^n}$

$\sum_{n=0}^{\infty} \frac{2}{10^n}$ geom. series $r = \frac{1}{10}$, $|r|<1$

$= \sum_{n=0}^{\infty} \frac{2}{10}\left(\frac{1}{10}\right)^{n-1}$ convergent.

$\therefore \sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$ is convergent by Comp Test

#26 $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$

$a_n = \frac{n+5}{\sqrt[3]{n^7+n^2}}$, $b_n = \frac{n}{\sqrt[3]{n^7}} = \frac{n^1}{n^{7/3}} = \frac{1}{n^{4/3}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is convergent. p-series $p = 4/3 > 1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+5}{(n^7+n^2)^{1/3}} \cdot \frac{n^{4/3}}{1}$$

$$= \lim_{n\to\infty} \left(\frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}}\right) \frac{1/n^{7/3}}{\frac{1}{n^{7/3}}} = \lim \frac{1+5/n}{\left((n^7+n^2)\left(\frac{1}{n^7}\right)\right)^{1/3}}$$

$$= \lim_{n\to\infty} \frac{1+\underbrace{5/n}_{\to 0}}{(1+\underbrace{1/n^5}_{\to 0})^{1/3}} = \frac{1}{1} = 1$$

$\therefore$ convergent by limit comp. Test.

<u>OR</u> Comparison Test

$$0 \le \frac{n+5}{\sqrt[3]{n^7+n^2}} \le \frac{n+5n}{\sqrt[3]{n^7}} = \frac{6n}{n^{7/3}}$$

$$= \frac{6}{n^{4/3}}$$

$\sum \frac{6}{n^{4/3}}$ conv.
$p = 4/3 > 1$
p-series

So convergent
by Comparison Test.