Next Thurs. Test §11.1-11.5

<u>Today</u> §11.5 The Alternating Series Test

Hw §11.5 #1-34

<u>Example</u>: The following series are called <u>alternating series</u>.

- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$
- $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} + \dots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$

We write a general form as

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n} b_n$$

where $b_n > 0$.

We have the following test to determine whether an alternating series is convergent or divergent.

The Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots$$

$$b_n > 0$$

satisfies

(i) $b_{n+1} \le b_n$ for all n

i.e. b_n is decreasing

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

EXAMPLE: Determine whether the following series is convergent.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

SOLUTION

We use the Alternating Series Test.

$b_n=\frac{1}{n}>0$

(i) Show b_n is decreasing.

$$b_{n+1}\le b_n$$

$$\frac{1}{n+1}\le\frac{1}{n}$$

$$n\le n+1$$

$$0\le 1 \quad \checkmark$$

(ii) $\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty}\frac{1}{n}=0 \quad \checkmark$ ⇐ If this fails, ($\lim b_n\neq 0$) then the series is divergent by the Test for Divergence. (§11.2 p.692)

Therefore by A.S.T. the series converges.

EXAMPLE Is the following series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} = \frac{-3}{3} + \frac{6}{7} - \frac{9}{11} + \frac{12}{15} - \cdots$$

$$b_n = \frac{3n}{4n-1} > 0$$

(i) $\lim_{n\to\infty} b_n$

$$= \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$$

The series is divergent by the Test for Divergence.

Proof of Alternating Series Test.

We have $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6.$

assume $b_n > 0$

(i) $b_{n+1} \le b_n$

(ii) $\lim_{n\to\infty} b_n = 0$

Show the series converges.

$S_1 = b_1$

$S_2 = b_1 - b_2$

$S_3 = b_1 - b_2 + b_3$

$S_4 = (b_1 - b_2) + b_3 - b_4$

$S_5 = b_1 - b_2 + b_3 - b_4 + b_5$

$S_6 = b_1 - b_2 + b_3 - b_4 + b_5 - b_6$

Let's only look at the evens

$S_2 = b_1 - b_2 \ge 0$ because $b_{n+1} \le b_n$

$S_4 = \underset{\uparrow \text{pos}}{S_2} + \underset{\uparrow \text{pos}}{(b_3 - b_4)} \ge S_2$

$S_{2n} = S_{2n-2} + \underbrace{(b_{2n-1} - b_{2n})}_{\text{pos}} \ge S_{2n-2}$

We have

$0 \le S_2 \le S_4 \le S_6 \le \ldots \le S_{2n-2} \le S_{2n}$

So S_{2n} is increasing

We can write

$S_{2n} = b_1 - (\overbrace{b_2 - b_3}^{\text{pos}}) - (\overbrace{b_4 - b_5}^{\text{pos}}) - \ldots - (\overbrace{b_{2n-2} - b_{2n-1}}^{\text{pos}}) - \overset{\text{pos}}{b_{2n}}$

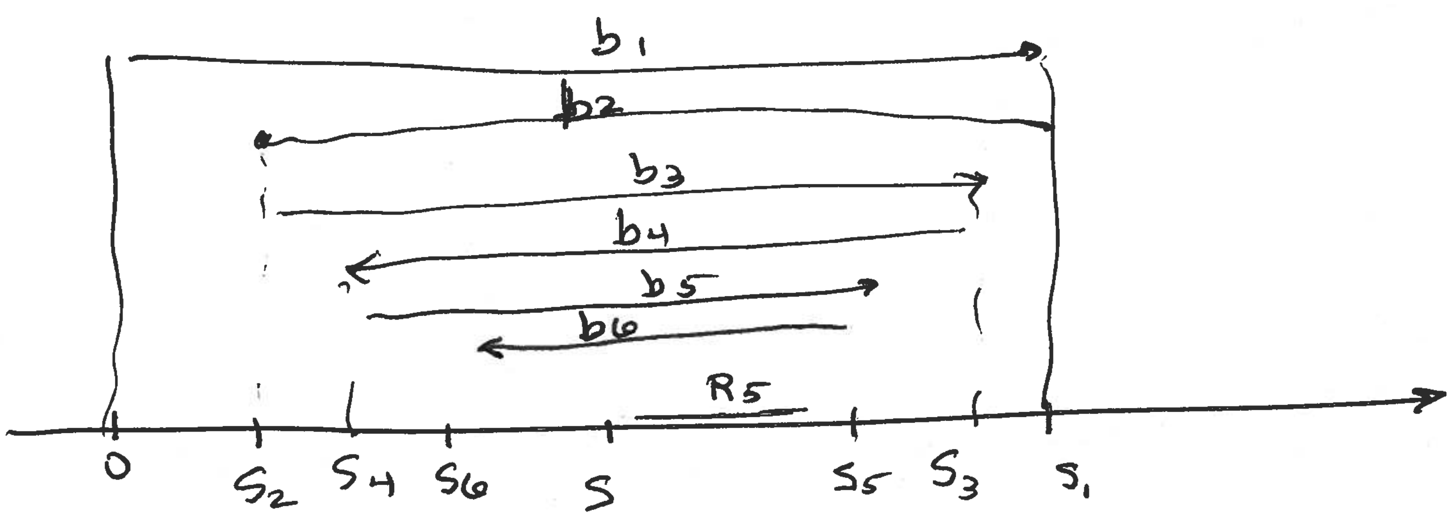
$b_3 \le b_2$

$\le b_1$

So S_{2n} is bounded and increasing. By the Monotonic Convergence Theorem, it ~~has~~ ~~limit~~ converges.

$$\lim_{n\to\infty} S_{2n} = S$$

Also $$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + b_{2n+1})$$
$$= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1}$$
$$= S + 0 = S.$$



Follow up : The Remainder is

$$R_n = S - S_n$$

we see that $$|R_n| \le b_{n+1}$$

Alternating Series Remainder Theorem

If $s = \sum (-1)^{n-1} b_n$ is an alternating series with $b_n > 0$

(i) $0 \leq b_{n+1} \leq b_n$ (ii) $\lim_{n \to \infty} b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$

EXAMPLE Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

SOLUTION First show it satisfies the alternating series test.

$$b_n = \frac{1}{n!} > 0$$

(i) $b_{n+1} \leq b_n$

$$\frac{1}{(n+1)!} \leq \frac{1}{n!}$$

$$n! \leq (n+1)!$$

$$n! \leq (n+1)n(n-1)\cdots 3\cdot 2\cdot 1$$

$$n! \leq (n+1)n!$$

$$1 \leq n+1$$

$$0 \leq n \; \checkmark$$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!}$

$$0 \leq \frac{1}{n!} \leq \frac{1}{n}$$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$ $\therefore$ $\lim_{n\to\infty} \frac{1}{n!} = 0$

(Sqz Theorem)

$$S = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

Note

$$b_7 = \frac{1}{7!} = \frac{1}{5040} < \frac{1}{5000} = 0.0002$$

$$|R_6| \leq b_7 \leq 0.0002$$

It is sufficient to find S_6 to approximate S to three decimal places

$S_6 \approx 0.368056$

$S \approx 0.368$ accurate to 3 decimal places.