§11.4 The Comparison Test

§11.4 #1-31 odd

<u>Example</u>: Is the series convergent or divergent?

p-series

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ~~conv~~ p-series
$p = 2 > 1$
convergent

b) $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series
divergent.

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series
$p = \frac{1}{2} \le 1$
divergent

d) $\sum \frac{1}{n^p}$ convergent for $p > 1$
divergent $p \le 1$

Geom. series

e) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$ Geom. series
$r = 1/2$
$|r| < 1$ convergent

f) $\sum_{n=1}^{\infty} \left(\frac{\pi}{3}\right)^n$ Geom. series
$r = \frac{\pi}{3} > 1$ divergent.

$= \sum_{n=1}^{\infty} \frac{\pi}{3}\left(\frac{\pi}{3}\right)^{n-1}$ $a = \frac{\pi}{3}$

g) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$\frac{1}{5} < \frac{1}{4}$ $5>4$

- $0 \le \frac{1}{2^n+1} \le \frac{1}{2^n}$

$\cancel{\sum_{n=1}^{10} \frac{1}{2^n+1} < \sum_{n=1}^{10} \frac{1}{2^n}}$ ← Scratch (unnecessary)

- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent because Geom. series, $r = \frac{1}{2}$

~~Does this inequality hold for infinite sums. Yes.~~

So $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent by the Comparison Test

h) $\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^3}$

- $0 \le \frac{1}{n} = \frac{n^2}{n^3} \le \frac{n^2+n+5}{n^3}$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent because it is a harmonic series (p-series, $p=1$)

$\therefore$ by Comparison Test $\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^3}$ is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series positive terms.

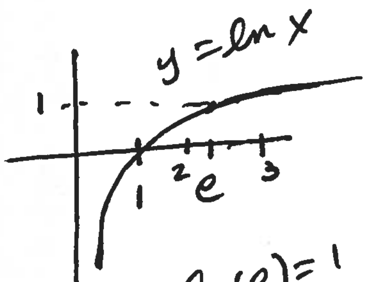
(i) If $\sum b_n$ is convergent and $a_n \le b_n$, for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $b_n \le a_n$ for all n, then $\sum a_n$ is also divergent.

Example: Determine whether the series is divergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

- $0 \le \frac{1}{n} \le \frac{\ln n}{n}$ $\quad n \geq 3$
- $\sum \frac{1}{n}$ is divergent
  $p=1$, p-series
  harmonic series



$\ln(e) = 1$

$\therefore$ $\sum \frac{\ln n}{n}$ is divergent by the comparison.

EXAMPLE: Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$$

<u>SOLUTION</u> We use the limit comparison test.

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$$

$$b_n = \frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^{5/2}} = \frac{1}{n^{1/2}}$$

The rule of thumb is to throw away the lower degree terms.

$\sum b_n = \sum \frac{1}{n^{1/2}}$ is divergent, p-series, $p = \frac{1}{2} < 1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2n^2+3n}{\sqrt{5+n^5}}\right) \Big/ \left(\frac{1}{n^{1/2}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2n^2+3n}{\sqrt{5+n^5}}\right)\left(\frac{n^{1/2}}{1}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}}\right) \frac{1/n^{5/2}}{1/\sqrt{n^5}}$$

$$= \lim_{n\to\infty} \frac{2+\underbrace{3/n}_{\to 0}}{\sqrt{\underbrace{5/n^5}_{\to 0}+1}} = 2$$

The Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c>0$, then either both series are convergent or divergent.

EXAMPLE Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.

SOLUTION We use Limit Comparison Test.

- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, because Geom. series, $r=\frac{1}{2}$.

- Let $a_n = \frac{1}{2^n-1}$, $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\left(\frac{1}{2^n-1}\right)}{\lim_{n\to\infty}\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \left(\frac{1}{2^n-1}\right)\left(\frac{2^n}{1}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2^n}{2^n-1}\right)\frac{1/2^n}{1/2^n} = \lim_{n\to\infty} \frac{1}{1-\underbrace{1/2^n}_{\to 0}} = 1$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent by the Limit Comp test.

Therefore $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is divergent by the limit comparison test.

Proof of Limit Comp Test.

Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $c>0$, real number

There exist positive numbers m and M such that

$$m \le \frac{a_n}{b_n} \le M \quad \text{for } n \geq N$$

for some $N>0$

$\Rightarrow$ $m b_n \le a_n \le M b_n$

If $\sum b_n$ diverges, then $\sum m b_n$ diverges. By Comp Test $\sum a_n$ also diverges.

If $\sum b_n$ converges, then $\sum M b_n$ converges, so by Comp Test $\sum a_n$ converges. $\square$

~~§II~~ §11.2 # 31) $\sum_{n=1}^{\infty} \arctan(n)$

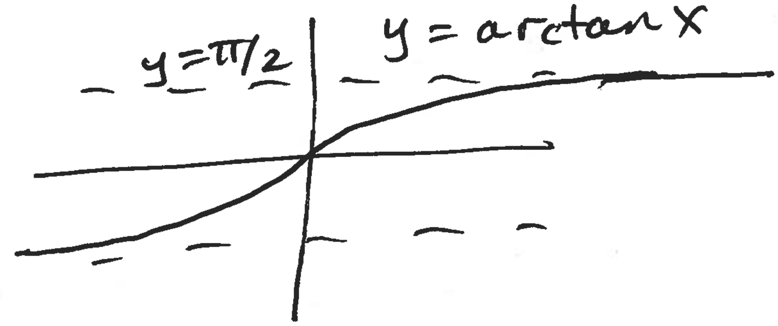
Is the series convergent or divergent?

Test for Divergence

If $\lim_{n\to\infty} a_n$ dne or $\lim_{n\to\infty} a_n \neq 0$,

then $\sum_{n=1}^{\infty} a_n$ is divergent.

$$\lim_{n\to\infty} \arctan(\underset{\nearrow \infty}{n}) = \pi/2$$



$\therefore$ $\sum_{n=1}^{\infty} \arctan(n)$ is divergent by the Test for Divergence.

§11.2 Express the number as a ratio of integers.

#42) $0.\overline{73} = 0.737373\ldots$

$= .73 + .0073 + .000073 + \ldots$

$= \frac{73}{100} + \frac{73}{10{,}000} + \frac{73}{1{,}000{,}000}$

$a = \frac{73}{100} \qquad r = \frac{1}{100}$

$$S = \frac{a}{1-r} = a\left(\frac{1}{1-r}\right)$$

$$= \frac{73}{100}\left(\frac{1}{1-1/100}\right)$$

$$= \frac{73}{100}\left(\frac{1}{99/100}\right)$$

$$= \frac{73}{100} \cdot \frac{100}{99} = \frac{73}{99}$$

Find the values of x for which the series converges. Find the sum of the series for those values of x.

\# 48) $\displaystyle\sum_{n=1}^{\infty} (x-4)^n$

$$= \sum_{n=1}^{\infty} (x-4)\cdot(x-4)^{n-1} \qquad \text{Geom Series.}$$

$a = x-4, \qquad r = x-4$

Convergent for

$|r| < 1$

$\boxed{|x-4| < 1}$

$-1 < x-4 < 1$

$\boxed{3 < x < 5}$

If $-1 < x < 5$, then the sum is

$$s = a\left(\frac{1}{1-r}\right) = (x-4)\frac{1}{1-(x-4)}$$

$$= \boxed{\frac{x-4}{-x+5}}$$