§11.2 Continued.

① $\sum_{n=1}^{\infty} c\,a_n = c\sum_{n=1}^{\infty} a_n$, c is a constant.

② $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

③ $\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

This is true for finite sums.

Example ① $\sum_{n=1}^{5} 3\cdot\left(\frac{1}{2}\right)^{n-1}$

$$= 3\left(\frac{1}{2}\right)^0 + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4$$

$$= 3\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right]$$

$$= 3\sum_{n=1}^{5}\left(\frac{1}{2}\right)^{n-1}$$

② $\sum_{n=1}^{3}\left(\frac{1}{n}+\frac{1}{2^n}\right) = \left(\frac{1}{1}+\frac{1}{2}\right) + \left(\frac{1}{2}+\frac{1}{2^2}\right) + \left(\frac{1}{3}+\frac{1}{2^3}\right)$

$$= \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right)$$

$$= \sum_{n=1}^{3}\frac{1}{n} + \sum_{n=1}^{3}\left(\frac{1}{2^n}\right)$$

§11.3 Continued

Estimating the Sum of a Series

We can not (in general) find the ~~exact~~ exact value for the sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 1$

(Recall this a p-series and is convergent for $p > 1$)

To find an approximation we can add up the first 10 terms or 100 terms. How many ~~should~~ terms should we add to get a good approximation.

EXAMPLE Approximate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms.

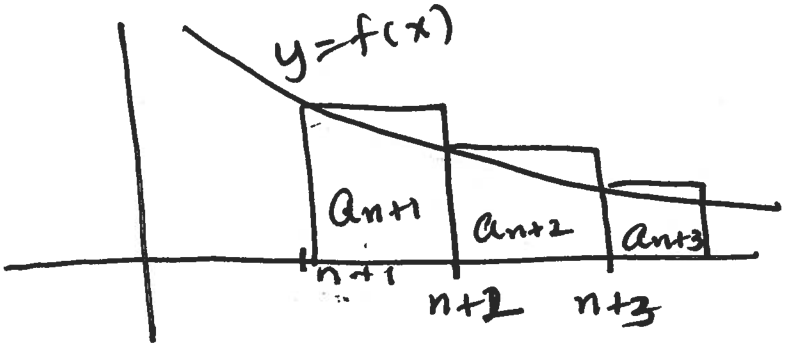
SOLUTION $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.1975$

Let $S = \sum_{n=1}^{\infty} a_n$, $\quad S_n = \sum_{i=1}^{n} a_i$

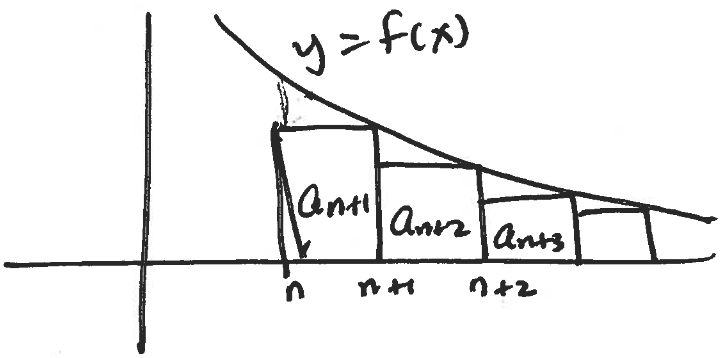
Define the remainder R_n as

$$R_n = S - S_n = (a_1 + a_2 + a_3 + \cdots) - (a_1 + a_2 + \cdots a_n$$

$$= a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$



a_n pos, decreasing, $f(x)$ contin.



$$\int_{n+1}^{\infty} f(x)\,dx \le a_{n+1} + a_{n+2} + \cdots \le \int_{n}^{\infty} f(x)\,dx$$

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

<u>Example</u>: For the last example we approximated $\sum_{n=1}^{\infty} \frac{1}{n^3}$

by summing the first 10 terms.

$S_{10} \approx 1.1975$

Estimate the error involved with this approximation.

$$0 \le R_n \le \int_n^{\infty} f(x)\,dx$$

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

$$= \lim_{t \to \infty} \int_{10}^{t} x^{-3}\,dx$$

$$= \lim_{t \to \infty} \left. \frac{x^{-2}}{-2} \right|_{10}^{t}$$

$$= \lim_{t \to \infty} \underbrace{\frac{-1}{2t^2}}_{\to 0} - -\frac{1}{2(10)^2}$$

$$= \frac{1}{2(10)^2} = \frac{1}{200}$$

$$= 0{,}005$$

Maximum error 0.005.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

$$R_n \le \int_n^{\infty} f(x)\,dx$$

$$R_n \le \int_n^{\infty} \frac{1}{x^3}\,dx \le 0.0005$$

Find n.

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty} \int_n^t x^{-3}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_n^t$$

$$= \lim_{t\to\infty} \left[\underbrace{\frac{-1}{2t^2}}_{\to 0} - -\frac{1}{2n^2}\right]$$

$$= \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < .0005$$

$$\frac{1}{2(.0005)} < n^2$$

$$\sqrt{\frac{1}{2(.0005)}} < n$$

$$n > 31.6$$

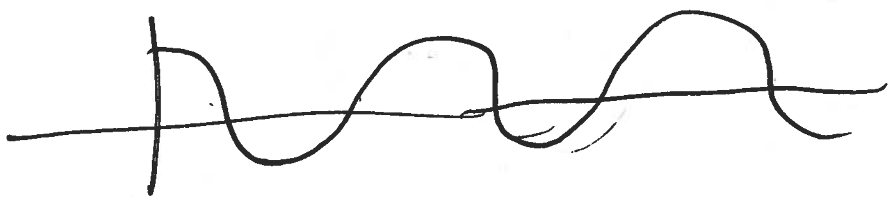
$$\boxed{n = 32}$$

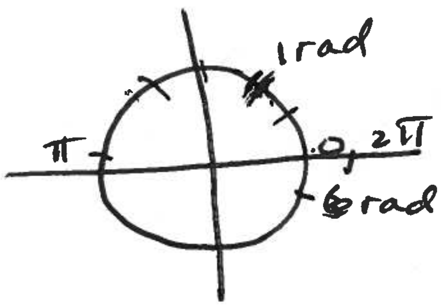
$\boxed{\text{HW § 11.3 \# 3-25 odd, 33}}$

§11.1 #27 $a_n = \cos\left(\frac{n}{2}\right)$

Diverges

$y = \cos\left(\frac{x}{2}\right)$





Follow up

$b_n = \cos\left(\frac{2}{n}\right)$ $\nearrow 0$

$\lim_{n \to \infty} b_n = \cos(0) = 1$

11.1 #37 $a_n = n \sin\left(\frac{1}{n}\right)$

$$a_n = \frac{\sin(1/n)}{(1/n)}$$

Method I $f(x) = \dfrac{\sin(1/x)}{1/x}$

$$\lim_{x\to\infty} \frac{\overset{\nearrow \sin 0 = 0}{\sin(1/x)}}{(1/x)_{\to 0}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\left(\cancel{-\frac{1}{x^2}}\right)}{\left(\cancel{-\frac{1}{x^2}}\right)}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$$

OR

$\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ by lemma from §3.3

Let $\theta = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)} = 1$$

§11.1 #39 $a_n = \left(1+\frac{2}{n}\right)^n$

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

$$= \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x} \quad \left(1+\frac{2}{x}\right)\to 1,\ x\to\infty$$

Use L'Hospital's Rule

1^∞

indet. power.

$$= \lim_{x\to\infty} e^{\boxed{\ln\left(1+\frac{2}{x}\right)^x}}$$

$$\lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x$$

$$= \lim_{x\to\infty} \underbrace{x}_{\to\infty} \underbrace{\ln\left(1+\frac{2}{x}\right)}_{\to \ln 1 = 0}$$

$\infty\cdot 0$ indet. product.

$$= \lim_{x\to\infty} \frac{\ln(1+2/x)}{(1/x)} \quad \begin{matrix}\to \ln 1 = 0\\ \to 0\end{matrix}$$

$\frac{0}{0}$ form

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{(1+2/x)}\cdot 2\left(\cancel{\frac{-1}{x^2}}\right)}{\cancel{\left(-1/x^2\right)}}$$

Aside

$$\left(\frac{1}{x}\right)' = (x^{-1})'$$
$$= -1\cdot x^{-2}$$
$$= -\frac{1}{x^2}$$

$$= \lim_{x\to\infty}\frac{2}{(1+2/x)} = \frac{2}{1+0} = 2 \quad (2/x \to 0)$$

Final Answer: e^2

§11.2 #19 $\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}}$

$r = \frac{\pi}{3} > 1$

divergent

§11.2 #25 $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{2^n}{3^n} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1}$$

$a=\frac{1}{3}$ $r=\frac{1}{3}$ $a=\frac{2}{3}$, $r=\frac{2}{3}$

both geom. series

$$\boxed{\underline{\text{FORMULA}}: \quad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for} \quad |r|<1}$$

$$= \frac{\frac{1}{3}}{(1-\frac{1}{3})} + \frac{\frac{2}{3}}{(1-\frac{2}{3})} = \frac{\left(\frac{1}{3}\right)\cdot 3}{\left(\frac{2}{3}\right)\cdot 3} + \frac{\left(\frac{2}{3}\right)\cdot 3}{\left(\frac{1}{3}\right)\cdot 3}$$

$$= \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$$

§11.1 #29 $\left\{ \frac{(2n-1)!}{(2n+1)!} \right\}$

Recall:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

$$(2n+1)! = (2n+1)(2n)(2n-1)(2n-2) \cdots 3 \cdot 2 \cdot 1$$

$$(2n-1)! = (2n-1)(2n-2) \cdots 3 \cdot 2 \cdot 1$$

$$a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)(2n-2) \cdots 3 \cdot 2 \cdot 1}{(2n+1)(2n)(2n-1)(2n-2) \cdots 3 \cdot 2 \cdot 1}$$

$$a_n = \frac{1}{(2n+1)(2n)}$$

$$\lim_{n \to \infty} \frac{1}{(2n+1)(2n)} = 0$$