§11.2 Continued

<u>Theorem</u> If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

<u>For example</u>: $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ is a convergent geom. series. $(a=1, r=\frac{1}{2})$

Therefore $\lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = 0.$

<u>Proof</u>: Let $S = \sum_{n=1}^{\infty} a_n$, where $\sum_{n=1}^{\infty} a_n$ is convergent.

~~If S_n are~~ If $S_n = \sum_{K=1}^{n} a_K$ are the partial sums, then $\lim_{n\to\infty} S_n = S$.

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$
$$S_{n-1} = a_1 + a_2 + a_3 + \dots + a_{n-1}$$

$\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S$

$$S_n - S_{n-1} = a_n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0$$

The Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist

or if $\lim_{n\to\infty} a_n \neq 0$, then the

series $\sum_{n=1}^{\infty} a_n$ is divergent.

(A implies B) true
iff
(not B implies not A) true
contrapositive

Note: The converse of the theorem is not true.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

but $\lim_{n\to\infty} \frac{1}{n} = 0$.

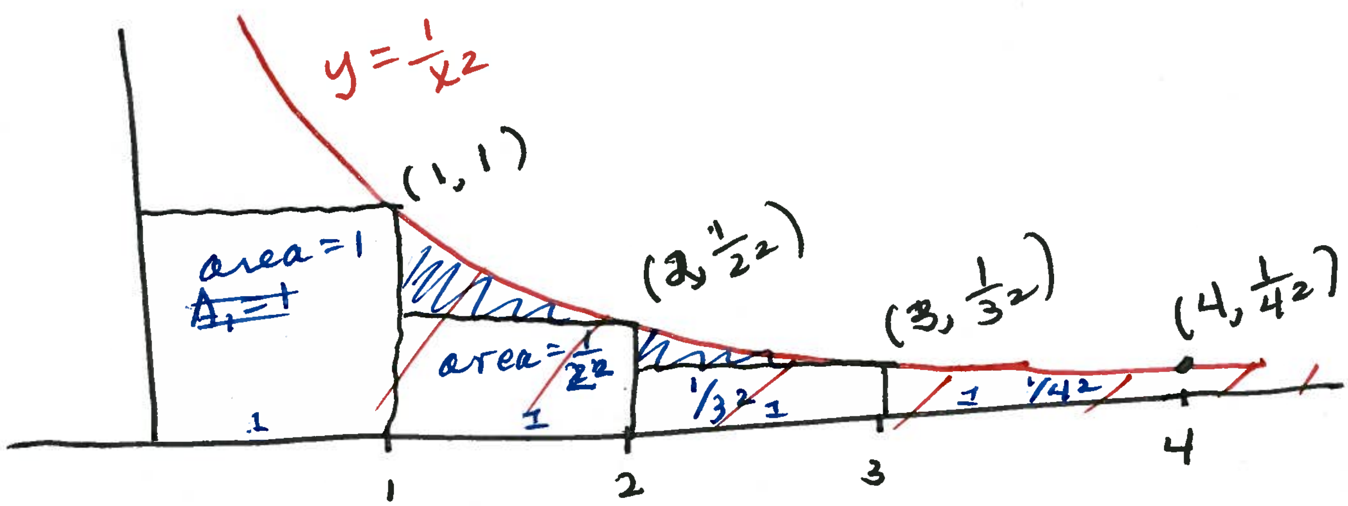
We will prove that
$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent
in §11.3.

§11.3 The Integral Test

HW §11.3 # 3–25 odd

EXAMPLE $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

This is not a geom. series or a telescoping series.



By the picture

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \leq 1 + \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}_{\text{area below the curve.}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t x^{-2}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^t$$

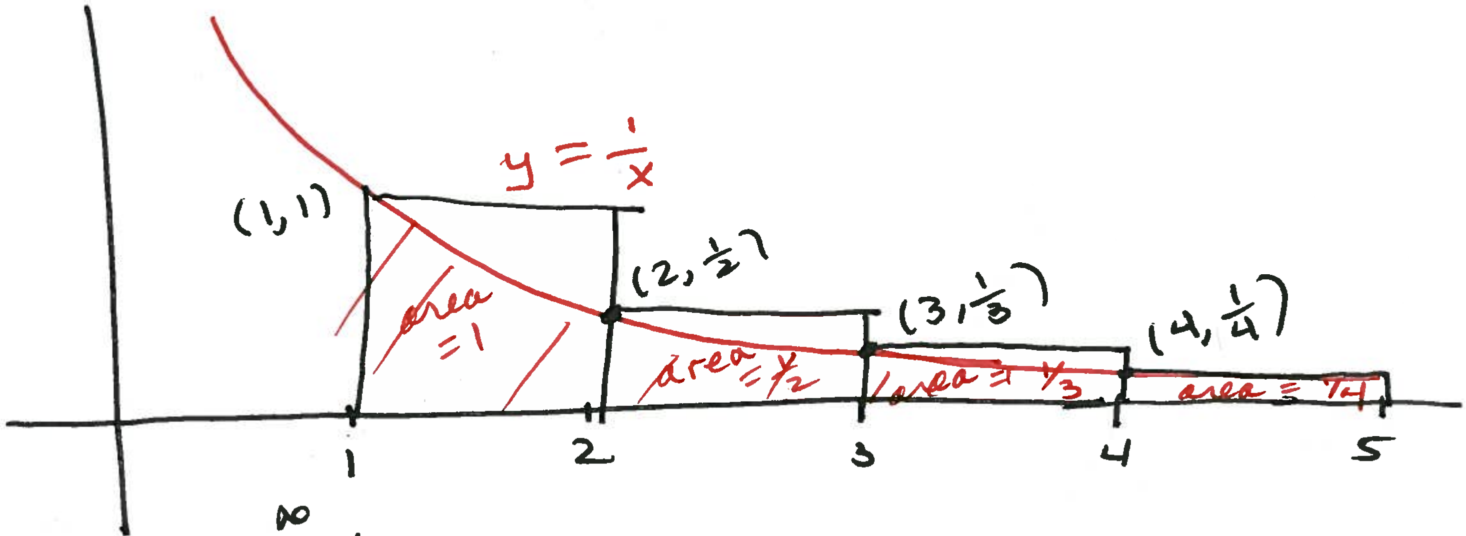
$$= \lim_{t\to\infty} \left[\overset{\nearrow 0}{-\frac{1}{t}} + 1\right] = 1$$

convergent.

Therefore $0 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1+1=2$

EXAMPLE

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Let's see why.



$$\int_1^{\infty} \frac{1}{x}\,dx \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

if $\int_1^{\infty} \frac{1}{x}\,dx$ is divergent, then

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ is divergent.

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t \to \infty} \Big[\ln|x|\Big]_1^t$$

$$= \lim_{t \to \infty} (\ln t - \ln 1) = \infty$$

divergent.

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ is called the harmonic series. It is divergent}}$

__The Integral Test__ Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if ~~the~~ $\int_1^{\infty} f(x)\,dx$ is convergent. That is,

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

~~Ex~~

Example: Is $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ convergent?

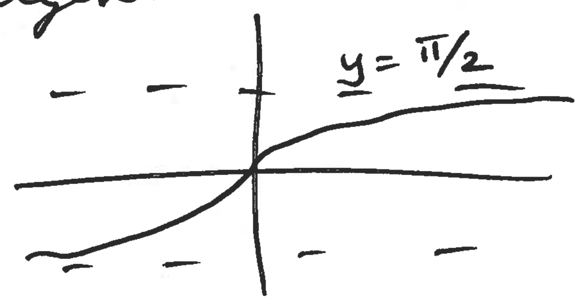
SOLUTION Let $f(x) = \frac{1}{x^2+1}$. f is decreasing, continuous, positive.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t \to \infty} \left[\tan^{-1} x\right]_1^t$$

$$= \lim_{t \to \infty} \left(\tan^{-1} t\right) - \left(\tan^{-1}(1)\right)$$

$$= \frac{\pi}{2} - 1 \quad \text{convergent.}$$



EXAMPLE For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution $\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent for $p>1$, divergent $p \le 1$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is called a p-series. It converges for } p>1, \text{ diverges for } p\le 1}$$

Example: Is the series convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$

p-series, $p=3$ so convergent.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$

p-series, $p=\frac{1}{2}$, divergent.

§ 11.1 # 25

$$a_n = \frac{(-1)^{n-1} n}{n^2+1}$$

$$|a_n| = \frac{n}{n^2+1}$$

$$\lim_{n \to \infty} \frac{n}{n^2+1} = \lim_{n \to \infty} \left(\frac{n}{n^2+1}\right) \frac{1/n^2}{1/n^2}$$

$$= \lim_{n \to \infty} \frac{1/n}{1 + 1/n^2} = \frac{0}{1} = 0$$

$\lim\limits_{n \to \infty} |a_n| = 0$ implies $\lim\limits_{n \to \infty} a_n = 0$

(Theorem 6)

11.1

#37 $\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right)$

$\theta = \frac{1}{n} \to 0$

$= \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)_{\to 0}} = 1$

Lemma: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

OR Use L'Hospital's Rule

$\lim_{x \to \infty} x^{\nearrow \infty} \sin\left(\frac{1}{x}\right)^{\nearrow \sin 0 = 0}$, $\frac{1}{x} \to 0$

$\infty \cdot 0$ form

$= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right) \to 0}{1/x \to 0}$

$\frac{0}{0}$ form

$\overset{L'H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}} = \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$

§ 11.2 # 33 $\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$

$$= \overset{\text{I}}{\sum_{n=1}^{\infty}\frac{1}{e^n}} + \overset{\text{II}}{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}}$$

I $\sum_{n=1}^{\infty}\frac{1}{e^n} = \sum_{n=1}^{\infty}\frac{1}{e}\left(\frac{1}{e}\right)^{n-1}$

geom. series. $a=\frac{1}{e}$, $r=\frac{1}{e}$

$$S=\frac{a}{1-r}=\left(\frac{1/e}{1-1/e}\right)\frac{e}{e}=\frac{1}{e-1}$$

II $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ ~~Telesco:~~ Telescoping Series.

$\cancel{\frac{1}{x(x+1)}}$ $\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$

$1=A(n+1)+Bn$

$n=0$

$1=A(0+1)$, $A=1$

$n=-1$

$1=B(-1)$, $B=-1$

$=\frac{1}{n}-\frac{1}{n+1}$

$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$S_1=\frac{1}{1}-\frac{1}{2}=1-\frac{1}{2}$

$S_2=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)=1-\frac{1}{3}$

$S_3=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=1-\frac{1}{4}$

$S_n=1-\frac{1}{n+1}$ $\quad s=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\overset{\to 0}{\frac{1}{n+1}}\right)=1$

I + II Answer $\frac{1}{e-1} + 1$

§11.1 #43

$\{0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1 \ldots\}$

divergent.

~~#63~~

§11.1 #65 Determine whether the series is increasing, decreasing or not monotonic. Is the sequence bounded?

$$a_n = \frac{n}{n^2+1}$$

I claim it is Decreasing

$$a_n \geq a_{n+1}$$

$$\frac{n}{n^2+1} \geq \frac{(n+1)}{(n+1)^2+1}$$

$$n\left[(n+1)^2+1\right] \geq (n+1)(n^2+1)$$

$$n\left[n^2+2n+1+1\right] \geq n^3+n+n^2+1$$

$$\cancel{n^3}+2n^2+2n \geq \cancel{n^3}+n^2+n+1$$

$$\phantom{\cancel{n^3}}-n^2 \quad -n \qquad\qquad -n^2 \quad -n$$

$$n^2+n \geq 1$$ yes, for $n \geq 1$

so decreasing.

<u>Bounded</u>:

$$0 \leq \frac{n}{n^2+1} \leq \frac{n^2+1}{n^2+1} = 1$$

Yes it is bounded.