§11.2 Continued.

A Geometric Series is of the form:

$$a + ar + ar^2 + ar^3 + ar^4 + \dots$$

$$= \sum_{n=1}^{\infty} a r^{n-1}$$

Example:

a) $3 + 3 \cdot \frac{1}{2} + 3 \cdot \left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \cdots$

$$= \sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1} \qquad a = 3, \; r = \frac{1}{2}$$

b) $3 + 3\left(-\frac{1}{2}\right) + 3\left(-\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right)^3 + \cdots$

$$= 3 - 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right)^3 + \cdots$$

$$= \sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^{n-1} \qquad a = 3, \; r = -\frac{1}{2}$$

$$= \sum_{n=1}^{\infty} 3(-1)^{n-1}\left(\frac{1}{2}\right)^{n-1}$$

For which values of r is the geometric series convergent?

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

Partial Sums: $S_1 = a$

$S_2 = a + ar$

$S_3 = a + ar + ar^2$

$S_n = a + ar + ar^2 + \ldots ar^{n-1}$

What is $\lim_{n\to\infty} S_n$?

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rS_n = \quad ar + ar^2 + ar^3 + \ldots \; ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-\overset{\to 0 \text{ if } |r|<1}{r^n})}{1-r}$$

$\lim_{n\to\infty} r^n = 0$ if $|r| < 1$ (that is $-1 < r < 1$)

So if $|r|<1$, then $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

$$\boxed{\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} \quad \text{for } |r|<1 \quad \text{(divergent otherwise)}}$$

Note: If $r=1$, the series is

$$\sum_{n=1}^{\infty} a r^{n-1} = a + a + a + \cdots$$

so this diverges.

if $r > 1$ or $r \leq -1$

then $\lim_{n\to\infty} r^n$ does not exist.

EXAMPLE: Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

SOLUTION a is always the first term.

$\boxed{a=5}$

$ar = -\frac{10}{3}$

$5r = -\frac{10}{3}$

$\boxed{r = -\frac{2}{3}}$

$ar^2 = 5\left(\frac{-2}{3}\right)^2 = \frac{20}{9}$

$$S = \sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^{n-1}$$

$$S = \frac{a}{1-r}$$

$$S = \frac{5}{1-(-2/3)} = \frac{5}{1+2/3} = \frac{5}{5/3}$$

$$= 5 \cdot \frac{3}{5} = 3$$

EXAMPLE: Write the number

$$2.3\overline{17} = 2.3171717\,17\ldots$$

as a ratio of integers.

$$S = 2.3 + .0171717171717\ldots$$

$$= 2.3 + .017 + .00017 + .0000017 + \ldots$$

$$= 2.3 + \left(\frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \frac{17}{10^9} + \ldots\right)$$

$$a = \frac{17}{10^3}, \quad r = \frac{1}{10^2}$$

$$= \frac{a}{1-r} = \frac{(17/10^3)}{1-\frac{1}{10^2}} \quad \frac{10^3}{10^3}$$

$$= \frac{17}{10^3 - 10} = \frac{17}{1000-10}$$

$$= \frac{17}{990}$$

$$S = 2.3 + \frac{17}{990} = \frac{99\;23}{99\;10} + \frac{17}{990}$$

$$= \frac{1147}{990}$$

<u>Example</u> : Write $0.\overline{9} = 0.9999\ldots$ as a fraction of integers.

Cheap Trick.

$$s = .\overline{9} = 0.9999$$

$$10s = 9.9999$$

$$10s - s = 9$$

$$\cancel{Sf} \quad 9s = 9$$

$$s = 1$$

Example: Find the sum of the series $\sum_{n=0}^{\infty} x^n$

$$= x^0 + x + x^2 + x^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

$$a = 1, \quad r = x, \qquad |r| < 1$$

$$S = \frac{a}{1-r} = \frac{1}{1-x}, \qquad |x| < 1$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

Example: Telescoping Series.

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Find the sum. (if it is convergent).

Partial Fractions

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$n=0$

$1 = A(0+1)$, $\boxed{A=1}$

$n=-1$

$1 = B(-1)$ $\boxed{B=-1}$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Partial Sums.

$S_1 = \frac{1}{1} - \frac{1}{1+1} = 1 - \frac{1}{2}$

$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$

$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$
$= 1 - \frac{1}{4}$

$S_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = 1 - \frac{1}{5}$

$S_n = 1 - \frac{1}{n+1}$ & $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)$
$= \boxed{1}$ $\square$