§11.1 Continued

Example

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6)$$

$n = 1, 2, 3, \ldots$

(a) Show $\{a_n\}$ is increasing.

Use induction.

Claim $a_{n+1} > a_n$.

Base case $n = 1$ show $a_{1+1} > a_1$

$a_2 > a_1$

$\frac{1}{2}(a_1 + 6) > 2$

$\frac{1}{2}(2+6) > 2$

$4 > 2$ ✓

~~Assume~~: Assume: $a_{k+1} > a_k$ Assume true for $n = k$

show $a_{k+2} > a_{k+1}$ show true for $n = k+1$

We have $a_{k+1} > a_k$

$+6 \quad +6$

$a_{k+1} + 6 > a_k + 6$

$\div 2 \quad \div 2$

$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$

$a_{k+2} > a_{k+1}$ ✓

(b) show $\{a_n\}$ is bounded.

$a_1 = 2$, ~~$a_{n+1} = \frac{1}{2}(a_n + 6)$~~

a_n is increasing ~~we see that~~

therefor $a_n \geq 2$ for ~~$0 \leq a_n$ for all n~~

all n.

Claim: $a_n \leq 6$ for all n.

Induction.

$\underline{n=1}$ $\quad a_1 = 2 < 6$ ✓

Assume true for $n=k$, show true for $n=k+1$

$$a_k \leq 6$$
$$+6 \quad +6$$

$$a_k + 6 \leq 12$$
$$\div 2 \quad \div 2$$

$$\underbrace{\tfrac{1}{2}(a_k+6)} \leq 6$$

$$a_{k+1} \leq 6 \quad ✓$$

Therefore $\{a_n\}$ is bounded and increasing. Therefore by the Monotonic Sequence Theorem, $\{a_n\}$ converges.

Let's Find the limit, L.

$$\lim_{n\to\infty} a_n = L$$

also $\lim_{n\to\infty} a_{n+1} = L$

$$a_{n+1} = \frac{1}{2}(a_n + 6)$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6)$$

$$L = \frac{1}{2}(L+6)$$

$$2L = L+6$$

$$\boxed{L=6}$$

Example: Show $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Recall:

$3! = 3\cdot 2\cdot 1$ $\quad 2! = 2\cdot 1$

$4! = 4\cdot 3\cdot 2\cdot 1$ $\quad 1! = 1$

$0! = 1$

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

$$a_n = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1}{\underbrace{n\cdot n\cdot n\cdot \cdots\cdots n\cdot n\cdot n}_{n \text{ times}}}$$

$$= \left(\cancel{\frac{n}{n}}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{3}{n}\right)\left(\frac{2}{n}\right)\left(\frac{1}{n}\right)$$

pos, less than 1

$$\le \frac{1}{n}$$

So $\quad 0 \le a_n \le \frac{1}{n}$

$$0 \le \frac{n!}{n^n} \le \frac{1}{n}$$

$\lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} \frac{1}{n} = 0$

So the Squeeze Theorem

$$\lim_{n\to\infty} a_n = 0$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Example Let $a_n = \frac{(-1)^n}{n}$, $n = 1, 2, 3, \ldots$

Find $\lim_{n\to\infty} a_n$.

~~So~~ **SOLUTION** $a_1 = \frac{(-1)^1}{1} = -1$

$a_2 = \frac{(-1)^2}{2} = \frac{1}{2}$

$a_3 = \frac{(-1)^3}{3} = \frac{-1}{3}$

$a_4 = \frac{(-1)^4}{4} = \frac{1}{4}$

We have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore $\lim_{n\to\infty} a_n = 0$.

Recall, we showed

if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

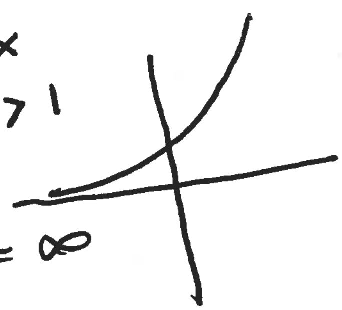
<u>Example</u> For which values of r is the sequence $\{r^n\}$ convergent?

$a_n = r^n$

Let's look at $f(x) = r^x$

$y = r^x$

$r > 1$



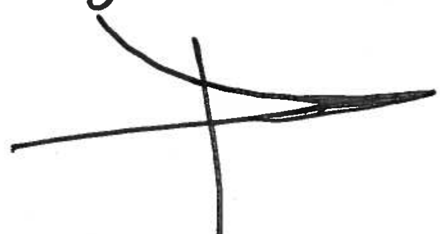
$\lim_{x\to\infty} r^x = \infty$

$y = r^x$

$0 < r < 1$

Example $r = \frac{1}{2}$

$y = \left(\frac{1}{2}\right)^x = 2^{-x}$



$\lim_{x\to\infty} 2^{-x} = 0$

we have

$\lim_{x\to\infty} r^x = 0$

if $0 < r < 1$.

If $r = 1$

$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n$

$= \lim_{n\to\infty} 1 = 1$

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 1 & \text{if } r = 1 \\ 0 & \text{if } 0 < r < 1 \\ 0 & \text{if } r = 0 \end{cases}$$

- $r = 0$

  $\lim_{n\to\infty} 0^n = 0$

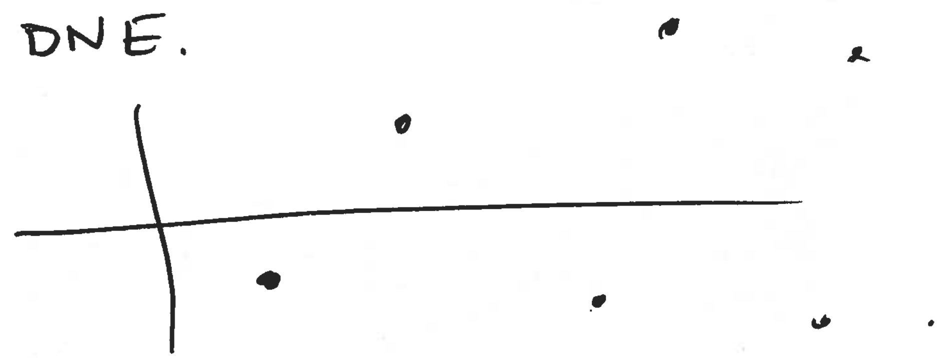
- $-1 < r < 0$ then $0 < |r| < 1$

  ~~then $|r|$~~

  $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$ if $0 < r < 1$

- If $r < -1$

  $\lim_{n\to\infty} r^n$ DNE.



Summary

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{Divergent otherwise.} \end{cases}$$

§11.2 Series.

HW §11.2 # 11-51 odd

__Example__ : The following are called series.

(a) ~~$1 + \frac{1}{2}$~~ $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

(b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

• Partial Sums.

Suppose we have a series

$a_1 + a_2 + a_3 + a_4 + \dots$

Define the partial sums as

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$S_N = a_1 + a_2 + a_3 + \dots + a_N$

The partial sums form a sequence.

$S_1, S_2, S_3, \dots$

that is

$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots$

If the sequence of partial sums converges, ~~then we~~

that is

$$\lim_{n\to\infty} S_n \text{ exists,}$$

then we say that the series converges.

We have

$$S_1 = a_1$$

$$S_2 = a_1 + a_2 = \sum_{n=1}^{2} a_n$$

$$S_3 = a_1 + a_2 + a_3 = \sum_{n=1}^{3} a_n$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \sum_{n=1}^{4} a_n$$

$$S_N = \sum_{n=1}^{N} a_n$$

~~We~~ If $\lim_{n\to\infty} S_N$ exists,

then we define

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_N$$

~~The~~ Example: Show that

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges. Find the sum. This is a geometric series.

SOLUTION

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \overset{n=1}{\frac{1}{2^{1-1}}} + \frac{1}{2^{2-1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4-1}} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Partial sums

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{2^2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$$

$$S_N = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^N}$$

Find $\lim_{N \to \infty} S_N$.

$$S_N = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^N}$$

$$\frac{1}{2} S_N = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^N} + \frac{1}{2^{N+1}}$$

$$S_N - \frac{1}{2} S_N = 1 - \frac{1}{2^{N+1}}$$

$$\frac{1}{2} S_N = 1 - \frac{1}{2^{N+1}}$$

$$S_N = 2\left(1 - \frac{1}{2^{N+1}}\right)$$

$$S_N = 2 - \frac{2}{2^{N+1}} \quad \frac{1\cancel{2}}{2^N \cdot \cancel{2}}$$

$$S_N = 2 - \frac{1}{2^N}$$

$$S_1 = 1$$

$$S_2 = 3/2$$

$$S_3 = 3/2 + \frac{1}{4} = \frac{7}{4}$$

$$S_4 = \frac{7}{4} + \frac{1}{16} = \frac{15}{16}, \quad S_5 = \frac{31}{32}$$

$$S_N = 2 - \frac{1}{2^N}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(2 - \overset{\nearrow 0}{\frac{1}{2^N}}\right) = 2$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

§10.4 # 15

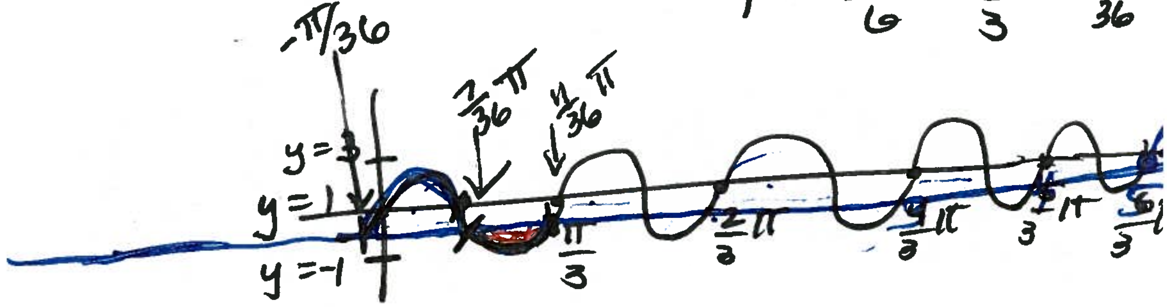
$r = 1 + 2\sin 6\theta$

Graph the curve and find the area that it encloses.

$y = 1 + 2\sin 6x$

$y = 2\sin 6x$

$P = \frac{2\pi}{6} = \frac{\pi}{3} = \frac{12\pi}{36}$



Find the zeros

$1 + 2\sin 6x = 0$

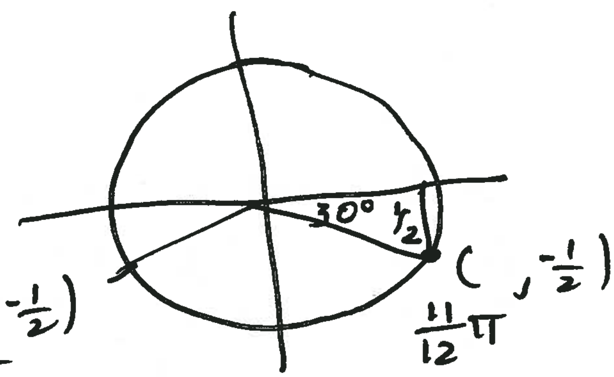
$2\sin 6x = -1$

$\sin 6x = -\frac{1}{2}$

$6x = \frac{7}{6}\pi + 2k\pi, \quad 6x = \frac{11}{6}\pi + 2k\pi$

$x = \frac{7}{36}\pi + \frac{2}{6}k\pi, \quad x = \frac{11}{36}\pi + \frac{2}{6}k\pi$

12 3.12   12 3.12 $(\ , -\frac{1}{2})$



$k=0, \quad x = \frac{7}{36}\pi, \ \frac{11}{36}\pi \qquad \frac{7}{6}\pi$

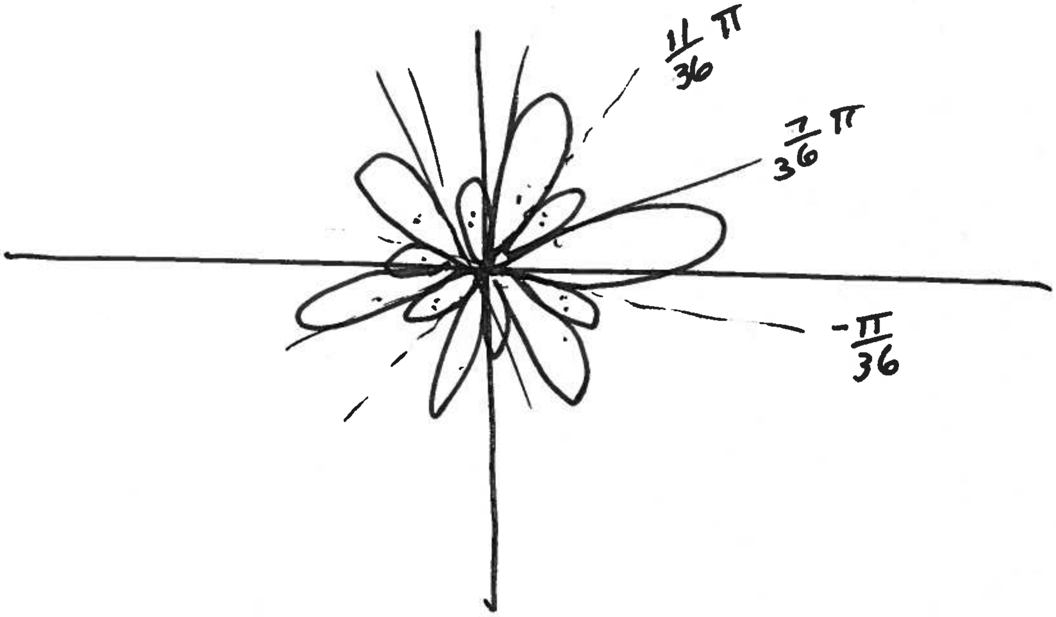
$k=1 \quad x = \frac{19}{36}\pi, \ x = \frac{23}{36}\pi$

$k=2$

$k=3$

$k=4$

$k=5, \quad x = \frac{67}{36}\pi, \ x = \frac{71}{36}\pi$



$\frac{11}{36}\pi$
$\frac{7}{36}\pi$
$-\frac{\pi}{36}$

§10.4 #21 $r = 1 + 2\sin\theta$ Find the area of the inner loop

Points of intersection

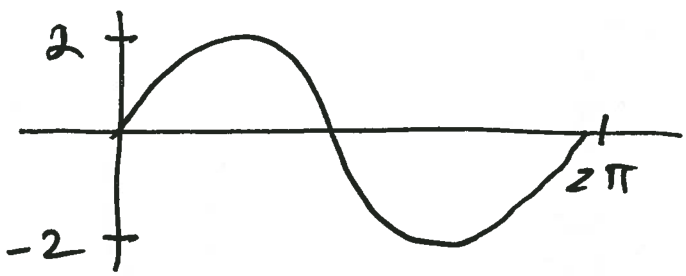
$1 + 2\sin\theta = 0$

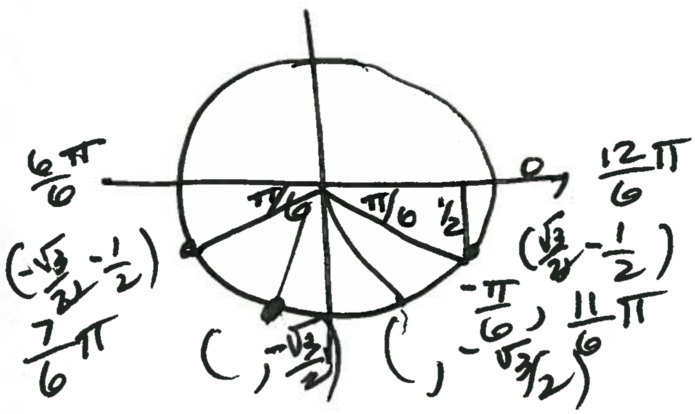
$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

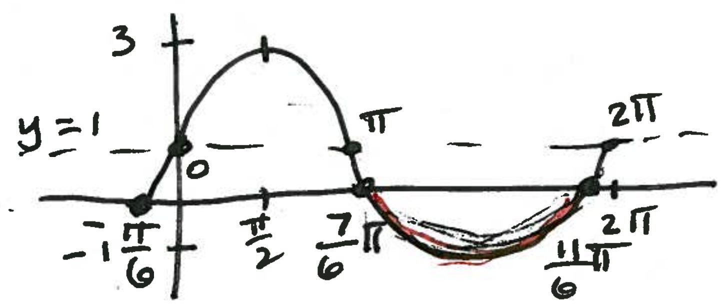
$\theta = -\frac{\pi}{6}, \ \frac{7}{6}\pi$

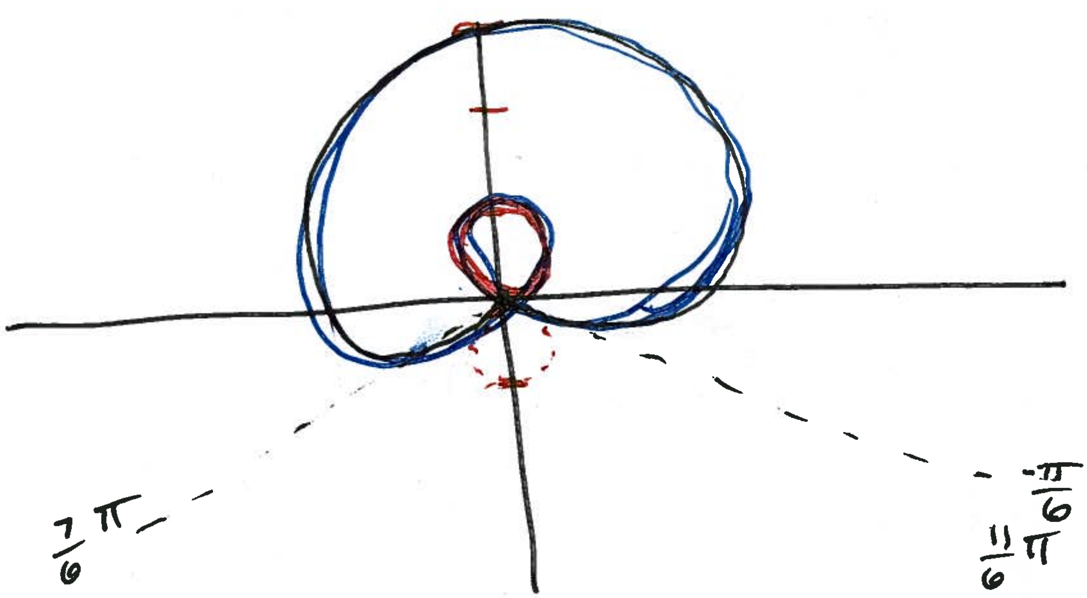
$y = 2\sin x$





$y = 1 + 2\sin x$





Inner Loop

$$A = \frac{1}{2}\int r^2\,d\theta$$

$$= \frac{1}{2}\int_{\frac{7}{6}\pi}^{\frac{11}{6}\pi} (1+2\sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{\frac{7}{6}\pi}^{\frac{11}{6}\pi} \left(1 + 2\cdot 2\sin\theta + 4\sin^2\theta\right)d\theta$$

$$= \frac{1}{2}\int_{\frac{7}{6}\pi}^{\frac{11}{6}\pi} \left(1 + 4\sin\theta + 4\cdot\frac{1}{2}(1-\cos 2\theta)\right)d\theta$$

$$= \frac{1}{2}\int_{\frac{7}{6}\pi}^{\frac{11}{6}\pi} \left(3 + 4\sin\theta - 2\cos 2\theta\right)d\theta$$

$$= \cancel{\frac{1}{2}\int_{\frac{7}{6}\pi}^{\frac{11}{6}\pi}} \quad \frac{1}{2}\Big[3\theta\Big]_{\frac{7}{6}\pi}^{\frac{11}{6}\pi} + -\frac{1}{2}4\Big[\cos\theta\Big]_{\frac{7}{6}\pi}^{\frac{11}{6}\pi}$$

$$+ \frac{1}{2}(-2)\left[\frac{1}{2}\sin 2\theta\right]_{\frac{7}{6}\pi}^{\frac{11}{6}\pi}$$

$\cos\frac{11}{6}\pi = \frac{\sqrt{3}}{2}$

$\cos\frac{7}{6}\pi = -\sqrt{3}/2$

$\sin 2\cdot\frac{11}{6}\pi = \sin\frac{11}{3}\pi = \sin\left(\underbrace{\frac{6}{3}\pi}_{=2\pi}+\frac{5}{3}\pi\right) = \sin\left(\frac{5}{3}\pi\right) = -\frac{\sqrt{3}}{2}$

$\sin 2\left(\frac{7}{6}\pi\right) = \sin\left(\frac{14}{6}\pi\right) = \sin\frac{7}{3}\pi = \sin\left(\frac{6}{3}+\frac{\pi}{3}\right)$

$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$= \frac{1}{2}\cdot 3\left(\frac{11}{6}\pi - \frac{7}{6}\pi\right) - 2\left[\cos\frac{11}{6}\pi - \cos\frac{7}{6}\pi\right]$$

$$- \frac{1}{2}\left(\sin 2\cdot\frac{11}{6}\pi - \sin 2\cdot\frac{7}{6}\pi\right)$$

$$= \frac{3}{2}\left(\frac{4}{6}\pi\right) - 2\left(\frac{\sqrt{3}}{2} + +\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\overbrace{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}^{-\sqrt{3}}\right)$$

$$= \pi - 2\sqrt{3} + \frac{1}{2}\sqrt{3} = \pi - \frac{3}{2}\sqrt{3}$$