Chapter 11 Sequences and Series

§11.1 Sequences

HW §11.1 #3–46, 60–66 ~~odd~~

EXAMPLE: ~~This is~~ These are examples of sequences.

a) 1, 3, 5, 7, 9, . . .

b) 1, 2, 4, 8, 16, . . .

c) 3, 1, 4, 1, 5, 9, 2, 6, 5, . . .

↑ expansion of π

A sequence is a function whose domain is the set of positive integers.

For (a) We have the sequence
1, 3, 5, 7, 9, . . . given by
the function ~~&~~: $a(n) = 2n - 1$.
$n = 1, 2, 3, 4, \ldots$

$a(1) = 2(1) - 1 = 1$
$a(2) = 2(2) - 1 = 3$
$a(3) = 2(3) - 1 = 5$,
etc.

We write
$a_n = 2n - 1$

The sequence $\{a_1, a_2, a_3, \ldots\}$ can be written

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

From the example (a),

$$\{1, 3, 5, 7, 9, \ldots\} = \{2n-1\}_{n=1}^{\infty}$$

~~where $a_n = 2n-1$~~

<u>EXAMPLE:</u> Write a formula for the general term an of the sequence

(a) $\underset{n=1}{1}, \underset{n=2}{\frac{1}{2}}, \underset{n=3}{\frac{1}{4}}, \underset{n=4}{\frac{1}{8}}, \underset{n=5}{\frac{1}{16}}, \ldots$ with $\cdot\frac{1}{2}$ between terms — Geometric sequence

$$\frac{1}{2^0}, \; 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$$

$r = \frac{1}{2}$

$$\boxed{a_n = \frac{1}{2^{n-1}}, \quad n = 1, 2, 3, \ldots}$$

or

$$\boxed{a_n = \frac{1}{2^n}, \quad n = 0, 1, 2, 3,}$$

(b) 3, 7, 11, 15, 19, 23, ... Arithmetic sequence.

$\overset{+4}{\curvearrowright}$ $\overset{+4}{\curvearrowright}$

$d = 4$

~~$a_n = n + 4$~~

$$3,\ 3+4,\ 3+4+4,\ 3+4+4+4, \ldots$$

$$3,\ 3+4,\ 3+2\cdot 4,\ 3+3\cdot 4,\ 3+4\cdot 4$$

$$a_n = 3 + 4n, \quad n = 0, 1, 2, 3, 4, \ldots$$

Geometric Sequence

$$a, \xrightarrow{\times r} ar, \xrightarrow{\times r} ar^2, ar^3, \ldots$$

$$a_n = ar^n, \quad n = 0, 1, 2, \ldots$$

Arithmetic Sequence

$$a, \xrightarrow{+d} a+d, \xrightarrow{+d} a+2d, a+3d, \ldots$$

$$a_n = a + nd, \quad n = 0, 1, 2, 3, \ldots$$

$$a_n = a + (n-1)d, \quad n = 1, 2, \ldots$$

(c) $3, \xrightarrow{\times 2} 6, \xrightarrow{} 12, \xrightarrow{} 24, \xrightarrow{} 48, \ldots$ Geom.

$3, 3 \cdot 2, 3 \cdot 2^2, 3 \cdot 2^3, 3 \cdot 2^4$

$$\boxed{a_n = 3 \cdot 2^n}$$

$n = 0, 1, 2, .$

$r = 2$

$a = 3$

$$\boxed{\{3 \cdot 2^n\}_{n=0}^{\infty}}$$

(d) $-2, \xrightarrow{+3} 1, \xrightarrow{+3} 4, 7, 10, 13, \ldots$ Arithmetic Seq.

$$a_n = 3n + -2$$

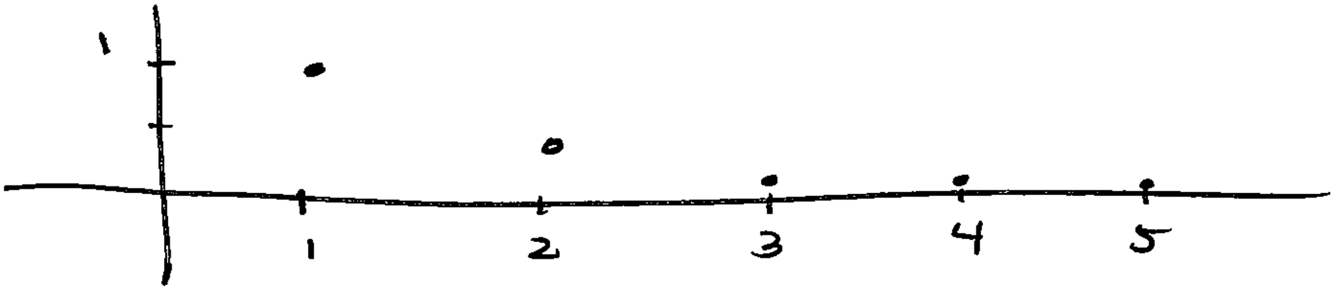
$n = 0, 1, 2, \ldots$

$$\{3n - 2\}_{n=0}^{\infty}$$

We can sketch the graph of a sequence.

<u>Example:</u> Plot points of the sequence.

(a) $a_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$



$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

We see that as $n \to \infty$, $a_n \to 0$. We write

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

If the limit of a sequence a_n as $n \to \infty$ is L, we write $\lim_{n \to \infty} a_n = L$.

The limit laws all apply to the limit of sequences.

The Sequeeze Theorem.

If $a_n \le b_n \le c_n$ for all $n \geq n_0$ (for some n_0), ~~then~~ and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof: Assume $\lim_{n\to\infty} |a_n| = 0$

we have

$$-|a_n| \le a_n \le |a_n|$$

$\lim_{n\to\infty} |a_n| = 0$ and $\lim_{n\to\infty} -|a_n| = 0$

so by Squeeze Thm

$$\lim_{n\to\infty} a_n = 0$$

Theorem: If $r > 0$, then

~~$\lim \frac{1}{x^a}$~~ $\lim_{n \to \infty} \frac{1}{n^r} = 0.$

Example: Find the limit.

a) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = \lim_{n \to \infty} \frac{1}{n^{1/2}}$

$= 0$

b) $\lim_{n \to \infty} \frac{2n^2+1}{n^2+3n+5}$

$= \lim_{n \to \infty} \left(\frac{2n^2+1}{n^2+3n+5}\right) \frac{1/n^2}{1/n^2}$

$= \lim_{n \to \infty} \frac{2 + 1/n^2}{1 + 3/n + 5/n^2} = \frac{2+0}{1+0+0} = 2$

We can sometimes use L'Hospital's Rule ~~(someti~~ to calculate limits of sequences, Example: Find $\lim\limits_{n\to\infty} \frac{\ln n}{n}$.

solution

~~We have~~ Let's find

$$\lim_{x\to\infty} \frac{\ln x^{\;\to\infty}}{x_{\;\to\infty}} \qquad \text{Indet. Form } \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Therefore $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$.

~~If [illegible]~~ If $y=f(x)$ is ~~[illegible]~~ a real valued function such that $f(n)=a_n$ for $n=1, 2, 3, \ldots$, Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{n\to\infty} a_n$.

special Case

- When $\theta = \pi$, $\frac{dy}{d\theta} = 0$, $\frac{dx}{d\theta} = 0$

so $\frac{dy}{dx} = \frac{0}{0}$ which is ~~an indeterim~~ undefined

Find

$$\lim_{\theta \to \pi} \frac{dy}{dx} = \lim_{\theta \to \pi} \frac{-\overset{\nearrow 0}{\sin^2\theta} + (1+\overset{\nearrow -1}{\cos\theta})\cos\theta}{\underset{\downarrow 0}{-\sin\theta}(1+2\cos\theta)} \quad \frac{0}{0} \text{ form}$$

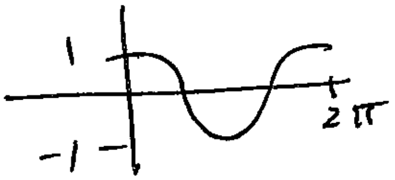
$$\underline{\underline{L'H}} = \lim_{\theta \to \pi} \frac{-2\overset{\nearrow 0}{\sin\theta}\cos\theta + (-\overset{\nearrow 0}{\sin\theta})\cos\theta + (1+\cos\theta)(\overset{\nearrow 0}{-\sin\theta})}{\underset{\downarrow}{-\cos\theta}\,(\underset{\downarrow}{1+2\cos\theta}) + (\underset{\downarrow}{-\sin\theta})(-2\cos\theta)}$$

$\cos\pi = -1$ $\quad 1+2\cos\pi$, $1+2(-1) = -1$ $\quad$ ~~$\sin\theta$~~ $\sin\pi = 0$ $\quad \to 1$

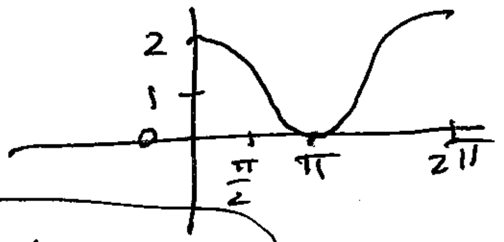
$= \frac{0}{1} = 0$ at $\theta = \pi$ horiz tangent line.

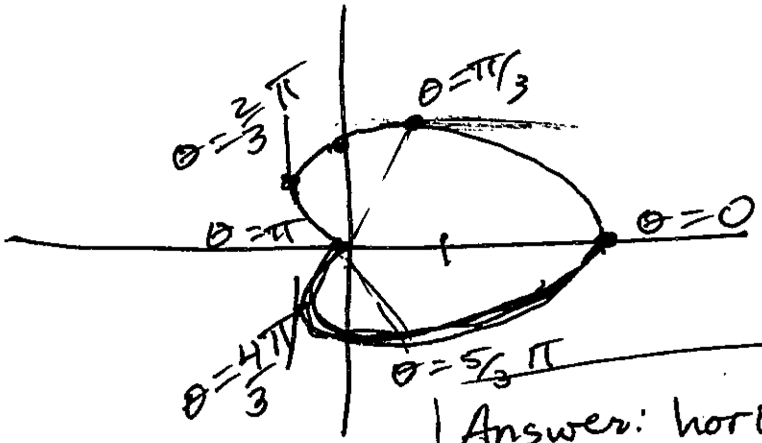
Sketch the graph $r = 1+\cos\theta$

$y = \cos x$



$y = 1+\cos x$





Answer: horiz tangent line:
$\theta = \pi/3, \pi, 5/3\,\pi$
Vertical Tangent line
$\theta = \frac{2}{3}\pi, \frac{4}{3}\pi, 0$

When is $\frac{dy}{d\theta} = 0$

$-\sin^2\theta + (1+\cos\theta)\cos\theta = 0$

$-\sin^2\theta + \cos\theta + \cos^2\theta = 0$

$-(1-\cos^2\theta) + \cos\theta + \cos^2\theta = 0$

$-1 + \cos^2\theta + \cos\theta + \cos^2\theta = 0$

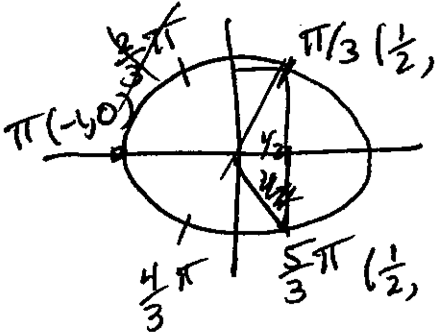
$u = \cos\theta$ $\quad 2\cos^2\theta + \cos\theta - 1 = 0$

$2u^2 + u - 1 = 0$

$(2u-1)(u+1) = 0$

$u = 1/2, \ u = -1$

$\cos\theta = 1/2 \quad \cos\theta = -1$



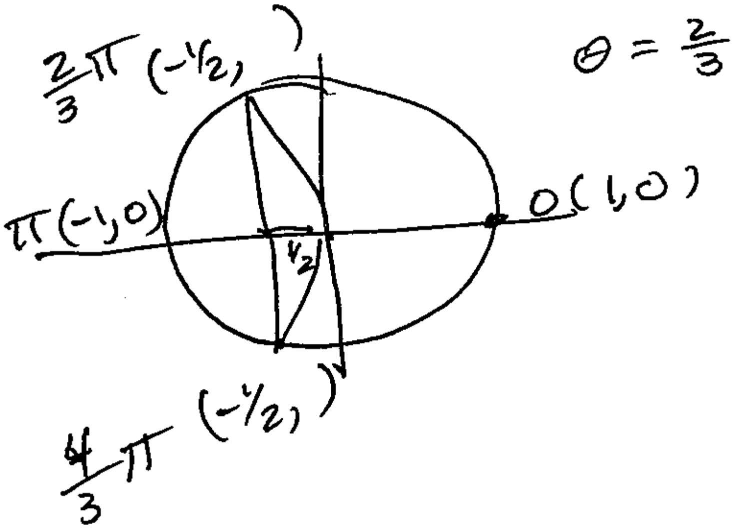
$\theta = \frac{\pi}{3}, \frac{5}{3}\pi, \pi$

Horiz. Tangent Line when $\frac{dy}{d\theta} = 0$

When is $\frac{dx}{d\theta} = 0$?

$\frac{dx}{d\theta} = -\sin\theta(1+2\cos\theta) = 0$

$\sin\theta = 0, \quad \cos\theta = \frac{-1}{2}$



$\theta = \frac{2}{3}\pi, \frac{4}{3}\pi, \pi, 0$

Vertical Tangent Line when $\frac{dx}{d\theta} = 0$.

§10.3 #65 Find where the tangent line is horizontal.
and ~~or~~ vertical

$r = 1 + \cos\theta.$

Solution:

Find where $\frac{dy}{dx} = 0.$

~~$y = xc$~~

$$x = r\cos\theta$$
$$x = (1+\cos\theta)\cos\theta$$
$$x = \cos\theta + \cos^2\theta$$
$$y = r\sin\theta$$
$$y = (1+\cos\theta)\sin\theta$$
$$y = \sin\theta + \cos\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

~~$\frac{dy}{d\theta} = \cos\theta$~~

$$y = (1+\cos\theta)\sin\theta$$
$$\frac{dy}{d\theta} = (-\sin\theta)\sin\theta + (1+\cos\theta)\cos\theta$$
$$x = (1+\cos\theta)\cos\theta$$
$$\frac{dx}{d\theta} = (-\sin\theta)\cos\theta + (1+\cos\theta)(-\sin$$
$$= -\sin\theta(1+2\cos\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{-\sin^2\theta + (1+\cos\theta)(\cos\theta)}{-\sin\theta(1+2\cos\theta)}$$