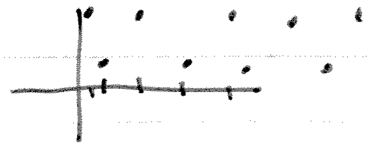
Math 185 Notes

Week 3

Weds. June 29, 2011

§11.1 #14

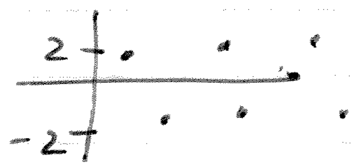
Find a formula for the nth term.



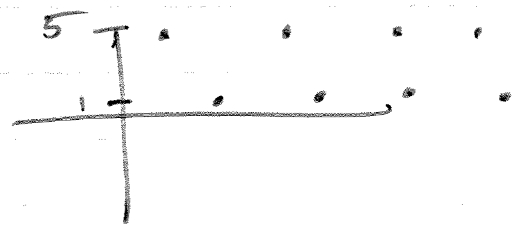
$\{5, 1, 5, 1, 5, 1, \ldots\}$

$\{(-1)^{n+1}\} = \{+1, -1, +1, -1, +1, -1, \ldots\}$

$\{(-1)^{n+1} \cdot 2\} = \{2, -2, 2, -2, 2, -2, \ldots\}$



$\{(-1)^{n+1} \cdot 2 + 3\} = \{5, +1, 5, +1, 5, 1, \ldots\}$



§11.1 # 21 Determine whether the sequence converges.

$$a_n = e^{1/n}$$

$$\lim_{n\to\infty} e^{(\overset{\nearrow 0}{1/n})} = e^0 = 1$$

<u>Aside</u> $\lim_{n\to\infty} \frac{1}{n} = 0$

#25 $a_n = \dfrac{(-1)^{n-1}\, n}{n^2+1} = (-1)^{n-1} \cdot \overset{\nearrow 0}{\dfrac{n}{n^2+1}}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| (-1)^{n-1} \frac{n}{n^2+1} \right| = \lim_{n\to\infty} \frac{n}{n^2+1}$$

$$= 0$$

$\therefore \lim_{n\to\infty} a_n = 0$

§11.1 #22 $a_n = \frac{3^{n+2}}{5^n} = \frac{3^n}{5^n} \cdot 3^2$

$= 9\left(\frac{3}{5}\right)^n$

Aside $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \text{diverges} & \text{o/w} \end{cases}$

otherwise

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 9\left(\frac{3}{5}\right)^n = 0$

§7.8 #26 $\displaystyle\int_0^\infty \frac{x \arctan x}{(1+x^2)^2}\,dx$

$$= \lim_{t\to\infty} \int_0^t \frac{x}{(1+x^2)^2} \arctan x\,dx$$

$u = \arctan x$ $\qquad$ $dv = \dfrac{x}{(1+x^2)^2}\,dx$

$du = \dfrac{1}{1+x^2}\,dx$ $\qquad$ $v = -\dfrac{1}{2(1+x^2)}$

$$= uv - \int v\,du$$

$$= \frac{-\arctan x}{2(1+x^2)} - \int \frac{-1}{2(1+x^2)} \cdot \frac{1}{1+x^2}\,dx$$

$$= \frac{-\arctan x}{2(1+x^2)} + \frac{1}{2}\boxed{\int \frac{1}{(1+x^2)^2}\,dx}$$

Aside $\displaystyle\int \frac{x}{(1+x^2)^2}\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\frac{1}{2}\int \frac{1}{u^2}\,du$$

$$= \frac{1}{2}\int u^{-2}\,du$$

$$= \frac{1}{2}\frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{2u} + C$$

$$= -\frac{1}{2(1+x^2)} + C$$

$$\int \frac{1}{(1+x^2)^2}\,dx \qquad x = \tan\theta$$
$$dx = \sec^2\theta\, d\theta$$

$$= \int \frac{1}{(1+\tan^2\theta)^2} \cdot \sec^2\theta\, d\theta$$

$$= \int \frac{1}{(\sec^2\theta)^2}\, \sec^2\theta\, d\theta$$

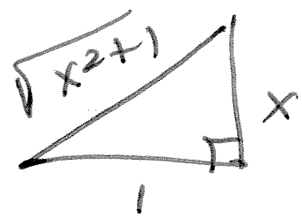
$$= \int \frac{1}{\sec^2\theta}\, d\theta = \int \cos^2\theta\, d\theta$$

$$= \int \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right)$$

$$\tan\theta = \frac{x}{1}$$

$$= \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\, 2 \cdot \sin\theta \cos\theta\right)$$



$$= \tfrac{1}{2}\left(\tan^{-1} x + \frac{x}{\sqrt{x^2+1}}\, \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \tfrac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)$$

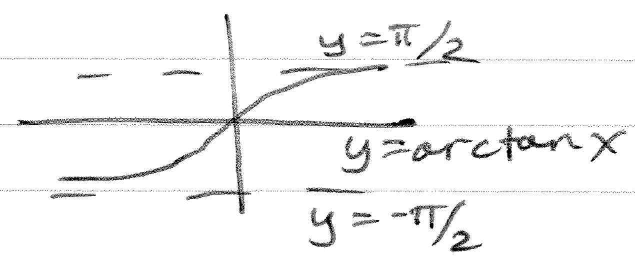
$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

$$\cos\theta = \frac{1}{\sqrt{x^2+1}}$$

$$= \lim_{t\to\infty} \left[\frac{-\arctan x}{2(1+x^2)} + \frac{1}{2}\cdot\frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)\right]_0^t$$

$$= \lim_{t\to\infty}\left[\overset{\to 0}{\frac{-\overset{\to \pi/2}{\arctan t}}{\underbrace{2(1+t^2)}_{\to\infty}}} + \frac{1}{4}\overset{\to\pi/2}{\arctan t} + \frac{1}{4}\overset{\to 0}{\frac{t}{t^2+1}}\right]$$

$$- \left[\frac{-\overset{=0}{\arctan 0}}{2(1+0^2)} + \frac{1}{4}\overset{=0}{\arctan 0} + \frac{1}{4}\frac{0}{0^2+1}\right]$$



$$= \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

Write in Wolfram alpha.com as

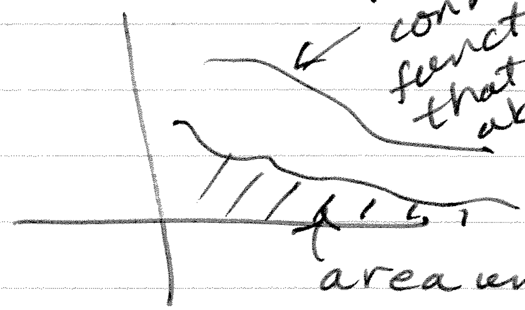
int_0^(infty)(x*atan(x))/(1+x^2)^2 dx

§7.8 EXAMPLE

Use the Comparison Test to show that the integral from §7.8 # 26 converges.

$$\int_0^\infty \frac{x \arctan x}{(1+x^2)^2}\, dx$$

find a convergent function that is above



area under curve. Show this area is finite

Step 1 Comparison

$x \geq 0$

$$0 \leq \frac{x \arctan x}{(1+x^2)^2} < \frac{x \quad \pi/2}{(1+x^2)^2}$$

$$< \frac{x}{(0+x^2)^2}\frac{\pi}{2}$$

$$= \frac{1}{x^3}\frac{\pi}{2}$$

Aside

$\int_a^\infty \frac{1}{x^p}\, dx$

$p > 1$ convergent

$p \leq 1$ divergent

Step 2

$$\frac{\pi}{2}\int_{1}^{\infty} \frac{1}{x^3}\, dx$$

convergent $\int_1^\infty \frac{1}{x^p} dx$, $p > 1$

Conclusion $\int_0^\infty \frac{x \arctan x}{(1+x^2)^2} dx = \int_0^1 \frac{x \arctan x}{(1+x^2)^2} dx + \int_1^\infty \frac{x \arctan x}{(1+x^2)^2} dx$

(*) Convergent by Comp. Test.

§11.1 (Continued)

<u>Definition</u> A sequence $\{a_n\}$ is called <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$. It is called <u>decreasing</u> if $a_n > a_{n+1}$ for all $n \geq 1$. It is called <u>monotonic</u> if it is either increasing or decreasing.

<u>Example</u>: Show that

$a_n = \frac{3}{n+5}$ is decreasing.

<u>SOLUTION 1</u>: $a_n > a_{n+1}$

$$\frac{3}{n+5} > \frac{3}{(n+1)+5}$$

$$3[(n+1)+5] > 3(n+5)$$

$$3(n+6) > 3(n+5)$$

$$n+6 > n+5$$

$$6 > 5 \checkmark$$

SOLUTION 2: $a_n = \frac{3}{n+5}$, $n \geq 1$

Let $f(x) = \frac{3}{x+5}$ $x \geq 1$

If I can show that f is decreasing, then it will follow that a_n is decreasing.

If $f'(x) < 0$ for $x \geq 1$, then f is decreasing.

$$f(x) = \frac{3}{(x+5)} = 3(x+5)^{-1}$$

$$f'(x) = 3(-1)(x+5)^{-2} = \frac{-3}{(x+5)^2} < 0$$

$\therefore$ f is decreasing which implies a_n is decreasing.

SOLUTION 3 Use induction to show that $a_n > a_{n+1}$.

$$a_n = \frac{3}{n+5}$$

- Base Case

$n=1$

$$a_1 > a_2$$

$$\frac{3}{1+5} > \frac{3}{2+5}$$

$$\frac{3}{6} > \frac{3}{7} \quad \text{yes.}$$

- Assume true for $n=k$. Show true for $n=k+1$.

Assume: $a_k > a_{k+1}$

$$\frac{3}{k+5} > \frac{3}{(k+1)+5}$$

$$3(k+6) > 3(k+5)$$

$$3k+18 > 3k+15$$

$$\quad +3 \qquad\quad +3$$

$$3k+21 > 3k+18$$

$$3(k+7) > 3(k+6)$$

$$\frac{3}{(k+1)+5} > \frac{3}{(k+2)+5}$$

$$a_{k+1} > a_{k+2}$$

Show $a_{(k+1)} > a_{(k+1)+1}$ ✓

$\square$

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \le M \quad \text{for all } n.$$

A sequence $\{a_n\}$ is bounded from below if there is a number M such that

$$M \le a_n \quad \text{for all } n.$$

If a sequence is bounded from above and below, then it is called bounded.

EXAMPLE • $a_n = \dfrac{n}{n+1}$ is bounded.

$$\{a_n\} = \Big\{ \underset{n=1}{\tfrac{1}{2}}, \underset{n=2}{\tfrac{2}{3}}, \underset{n=3}{\tfrac{3}{4}}, \underset{n=5}{\tfrac{4}{5}}, \ldots \Big\}$$

$$a_n = \frac{n}{n+1} = \frac{(n+1)-1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$\frac{1}{2} \le a_n \le 1$$ So $\{a_n\}$ is bounded

- $a_n = \frac{n}{n+1}$ is monotonic

a_n is increasing

$$f(x) = \frac{x}{x+1}$$

$$f'(x) = \frac{(x)'(x+1) - (x)(x+1)'}{(x+1)^2}$$

$$= \frac{(x+1) - x \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

$f'(x) > 0$ implies f is increasing

Therefore a_n is increasing.

So $\{a_n\}$ is monotonic.

By the Monotonic Sequence Theorem $\{a_n\} = \left\{\frac{n}{n+1}\right\}$ has a limit.

We see that $\lim_{n \to \infty} \frac{n}{n+1} = 1$.

§11.2 Series
HW 11.2 # 3-30

If we add the terms of a sequence we get

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots$$

We write

$$\sum_{n=1}^{\infty} a_n$$

uppercase sigma Σ

EXAMPLE A Geometric Series

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^{i-1} = \left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Partial Sums

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 7/4$

$S_4 = \underbrace{1 + \frac{1}{2} + \frac{1}{4}}_{7/4} + \frac{1}{8} = \frac{7}{4} + \frac{1}{8} = \frac{14}{8} + \frac{1}{8} = \frac{15}{8}$

$S_5 = \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}}_{15/8} + \frac{1}{16} = \frac{15}{8} + \frac{1}{16} = \frac{30}{16} + \frac{1}{16}$

$= 31/16$

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$$S_5 = \frac{32}{16} - \frac{1}{16} = 2 - \frac{1}{16}$$

$$S_6 = 2 - \frac{1}{32}$$

$$S_7 = 2 - \frac{1}{64}$$

$$S_n = 2 - \frac{1}{2^{n-1}}$$

$$S_{1000} = 2 - \frac{1}{2^{1000-1}}$$

We see that $\lim_{n \to \infty} S_n$

$$= \lim_{n \to \infty} 2 - \frac{1}{2^{n-1}} = 2$$

So we define the sum as

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

Note: $0.\overline{9} = 0.9999\ldots = 1$

$= .9 + .09 + .009 + .0009 + \ldots$

$= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$

$= \frac{9}{10} + \frac{9}{10}\cdot\left(\frac{1}{10}\right) + \frac{9}{10}\left(\frac{1}{10^2}\right) + \frac{9}{10}\left(\frac{1}{10^3}\right) + \cdots$

$= \sum_{n=1}^{\infty} \frac{9}{10}\cdot\left(\frac{1}{10}\right)^{n-1}$

This is a geometric series. Later we will derive a formula to find its sum.

__Definition__ Given a series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots$$

let s_n denote the partial sum

$$s_n = a_1 + a_2 + \dots + a_n$$

$$s_n = \sum_{i=1}^{n} a_i$$

If the sequence $\{s_n\}$ is convergent

(Do you see that $\{s_1, s_2, s_3, \dots\}$ form a sequence?)

~~If the seq sequence of partial sums s_n conv.~~
~~then~~

and $\lim_{n \to \infty} s_n = s$ exists as a real number, then the series $\sum_{i=1}^{\infty} a_i$ converges

and we write $a_1 + a_2 + a_3 + \dots = s$

that is $\sum_{n=1}^{\infty} a_n = s$.

The number s is called the <u>sum</u> of the series. Otherwise, the series is <u>divergent</u>.

Geometric Series

$$a + ar + ar^2 + ar^3 + \dots ar^{n-1} + \dots$$

$$= \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0.$$

Let $s_n = a + ar + ar^2 + \dots + ar^{n-1}$ be the nth partial sum.

$$s_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

$$rs_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n$$

$$s_n(1-r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

We take the limit of S_n.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

Note

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{divergent} & \text{otherwise} \end{cases}$$

Note if $r=1$, then the ~~sequ~~ series is $a + a + a + a + \ldots$ which diverges.

$$\lim_{n\to\infty} S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r} \quad \text{provided } -1 < r < 1$$

diverges otherwise

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

is convergent if $|r|<1$

and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r|<1$$

if $|r| \geq 1$, then the series diverges.

<u>EXAMPLE</u> Find the sum of the geometric series

$r = \frac{-2}{3}$

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots \quad \text{times } r = -2/3$$

$$= 5 + 5\left(\frac{-2}{3}\right) + 5\left(\frac{-2}{3}\right)^2 + 5\left(\frac{-2}{3}\right)^3 + \cdots$$

$$= \sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^{n-1}$$

$a = 5, \quad r = -2/3,$

note
$|r| = \left|\frac{-2}{3}\right| < 1$
convergent

$$S = \frac{a}{1-r} = \frac{5}{1-(-2/3)} = \frac{5}{1+2/3}$$

$$= \frac{5}{\left(\frac{3}{3}+\frac{2}{3}\right)} = \frac{5}{(5/3)} = 5 \cdot \frac{3}{5} = 3$$