§7.3 #1 $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$

$x = 3\sec\theta$, $0 \le \theta < \frac{\pi}{2}$, or $\pi \le \theta < \frac{3}{2}\pi$

$dx = 3\sec\theta\tan\theta\,d\theta$

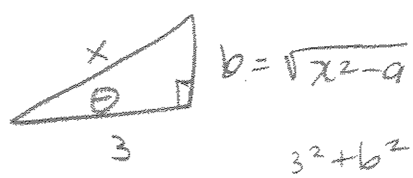
$$= \int \frac{1}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2-9}} \cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\sqrt{9(\sec^2\theta-1)}}\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\cdot 3\sqrt{\tan^2\theta}}\,d\theta = \int \frac{3\sec\theta\cancel{\tan\theta}}{9\sec^2\theta\cancel{\tan\theta}}\,d\theta$$

$$= \int \frac{1}{3}\frac{1}{\sec\theta}\,d\theta = \frac{1}{3}\int \cos\theta\,d\theta = \frac{1}{3}\sin\theta + C$$

$\sec\theta = \frac{x}{3} \Rightarrow \cos\theta = \frac{3}{x} = \frac{adj}{hyp}$



$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{x^2-9}}{x}$

$3^2 + b^2 = x^2$
$b^2 = x^2 - 9$
$b = \sqrt{x^2-9}$

Answer: $\frac{1}{3}\frac{\sqrt{x^2-9}}{x} + C$

§7.3 #2 $\int x^3\sqrt{9-x^2}\,dx$, $\quad x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$= \int (3\sin\theta)^3 \cdot 3\cos\theta \cdot 3\cos\theta\,d\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta}$

$= 3\sqrt{1-\sin^2\theta}$

$= 3\sqrt{\cos^2\theta} = 3\cos\theta$

$= 3^5 \int \sin^3\theta\cos^2\theta\,d\theta$

$= 243\int \sin^2\theta\cos^2\theta\sin\theta\,d\theta$

$= 243\int (1-\cos^2\theta)\cos^2\theta \cdot \sin\theta\,d\theta$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

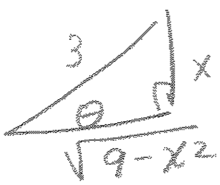
$-du = \sin\theta\,d\theta$

$= -243\int (1-u^2)u^2\,du$

$= -243\int (u^2-u^4)\,du = -243\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$

$= -243\left(\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right) + C$

$\sin\theta = \frac{x}{3} = \frac{\text{opp}}{\text{hyp}}$



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{9-x^2}}{3}$

$= -243\left(\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5\right) + C$

$= -3^5\left(\frac{1}{3^4}(9-x^2)^{3/2} - \frac{1}{5\cdot 3^5}(9-x^2)^{5/2}\right) + C$

$= \boxed{-3(9-x^2)^{3/2} + \frac{1}{5}(9-x^2)^{5/2} + C}$

§ 7.3 #3 $\quad \displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx; \qquad x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{(3\tan\theta)^3}{\cancel{3}\sec\theta}\,\cancel{3}\sec^2\theta\,d\theta$$

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9}$$

$$= \sqrt{9(\tan^2\theta+1)}$$

$$= \sqrt{9\sec^2\theta}$$

$$= 3\sec\theta$$

$$= \int 3^3\tan^3\theta\cdot\sec\theta\,d\theta$$

$$= 27\int \tan^2\theta\cdot\sec\theta\tan\theta\,d\theta$$

$$= 27\int (\sec^2\theta-1)\sec\theta\tan\theta\,d\theta$$

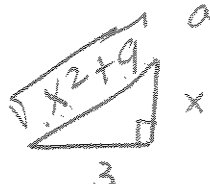
$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= 27\int (u^2-1)\,du$$

$$= 27\left[\frac{u^3}{3} - u\right] + C = 9\sec^3\theta - 27\sec\theta + C$$

$\dfrac{x}{3} = \tan\theta = \dfrac{\text{opp}}{\text{adj}}$



$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{\sqrt{x^2+9}}{3}$

$$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + C$$

$$= \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

§7.3 #4

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$$

$x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta}$

$= \sqrt{16(1-\sin^2\theta)}$

$= \sqrt{16\cos^2\theta}$

$= 4\cos\theta$

- when $x=0$

$4\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0$

- When $x = 2\sqrt{3}$,

$4\sin\theta = 2\sqrt{3}$

$\sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\theta = \pi/3$

$$\int_0^{\pi/3} \frac{(4\sin\theta)^3}{\cancel{4}\cos\theta}\,\cancel{4}\cos\theta\, d\theta$$

$$= 4^3 \int_0^{\pi/3} \sin^3\theta \cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

when $\theta = 0$, $u = \sin 0 = 0$

when $\theta = \pi/3$, $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$= 64 \int_0^{\sqrt{3}/2} u^3\, du$$

$$= 64 \cdot \left[\frac{u^4}{4}\right]_0^{\sqrt{3}/2}$$

$$= 16\left[\frac{\sqrt{3}}{2}\right]^4$$

$$= 16\left(\frac{3}{4}\right)^2 = 16\left(\frac{9}{16}\right) = 9$$

$\boxed{\text{Answer: } 9}$

⑤ $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$

$t = \sec\theta$

$dt = \sec\theta\tan\theta\,d\theta$

$\sqrt{t^2-1} = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\tan^2\theta} = \tan\theta$

- when $t = \sqrt{2}$,

$\sqrt{2} = \sec\theta$

$\frac{1}{\sqrt{2}} = \cos\theta$

$\frac{\sqrt{2}}{2} = \cos\theta, \quad \theta = \frac{\pi}{4}$

- when $t = 2$

$\sec\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$$\int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta \cdot \cancel{\tan\theta}} \cdot \sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/3} (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left[\theta\right]_{\pi/4}^{\pi/3} + \frac{1}{4}\left[\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] + \frac{1}{4}\left[\sin\frac{2}{3}\pi - \sin\Big(\underbrace{2\cdot\frac{\pi}{4}}_{=\pi/2}\Big)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{12}\right] + \frac{1}{4}\left[\sqrt{3}/2 - 1\right]$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

⑥ $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

$$\int_0^{\pi/3} \frac{\tan\theta}{\cancel{\sec\theta}} \cdot \cancel{\sec\theta}\tan\theta\, d\theta$$

$$= \int_0^{\pi/3} \tan^2\theta\, d\theta$$

$$= \int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$$

$$= \Big[\tan(\theta) - \theta\Big]_0^{\pi/3}$$

$$= \Big[\tan\theta\Big]_0^{\pi/3} - \Big[\theta\Big]_0^{\pi/3}$$

$$= \left(\tan\frac{\pi}{3} - \tan 0\right) - \left(\frac{\pi}{3} - 0\right)$$

$$= \sqrt{3} - 0 - \frac{\pi}{3}$$

$$= \boxed{\sqrt{3} - \frac{\pi}{3}}$$

$x = \sec\theta$

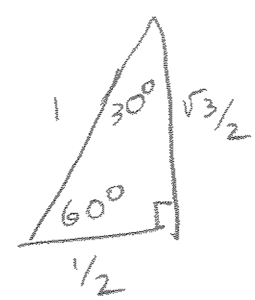
$dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\tan^2\theta}$

$= \tan\theta$

- When $x=1$
  $1 = \sec\theta$,
  $\cos\theta = 1$, $\theta = 0$
- When $x=2$
  $2 = \sec\theta$
  $\cos\theta = \frac{1}{2}$,
  $\theta = \pi/3$



$\tan 60° = \dfrac{\sqrt{3}/2}{1/2}$

$= \sqrt{3}$

§7.3 #7 $\quad \displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta}$

$= \sqrt{25}\,\sqrt{1-\sin^2\theta}$

$= 5\sqrt{\cos^2\theta}$

$= 5\cos\theta$

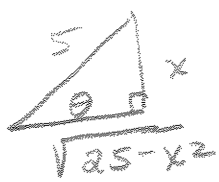
$$= \int \frac{1}{(5\sin\theta)^2 \cdot 5\cos\theta} \cdot 5\cos\theta\, d\theta$$

$$= \frac{1}{25}\int \frac{1}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{25}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{25}\cot\theta + C$$

$\frac{x}{5} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$



$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{25-x^2}}{x}$

$$= -\frac{1}{25}\,\frac{\sqrt{25-x^2}}{x} + C$$

§7.3 # 8 $\displaystyle\int \frac{x^3}{\sqrt{x^2+100}}\,dx$

$x = 10\tan\theta$

$dx = 10\sec^2\theta\,d\theta$

$\sqrt{x^2+100}$

$= \sqrt{100\tan^2\theta + 100}$

$= 10\sqrt{\tan^2\theta+1} = 10\sqrt{\sec^2\theta}$

$= 10\sec\theta$

$\displaystyle\int \frac{(10\tan\theta)^3}{\cancel{10}\sec\theta}\cdot \cancel{10}\sec^2\theta\,d\theta$

$= 1000\displaystyle\int \tan^3\theta\,\sec\theta\,d\theta$

$= 1000\displaystyle\int \tan^2\theta \cdot \sec\theta\tan\theta\,d\theta$

$= 1000\displaystyle\int (\sec^2\theta - 1)\cdot \sec\theta\tan\theta\,d\theta$

$u = \sec\theta$

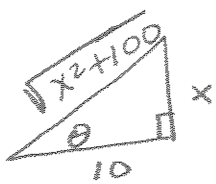
$du = \sec\theta\tan\theta\,d\theta$

$= 1000\displaystyle\int (u^2-1)\,du$

$= 1000\left(\frac{u^3}{3} - u\right) + C$

$= 1000\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$

$\frac{x}{10} = \tan\theta = \frac{opp}{adj}$



$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+100}}{10}$

$= 1000\left(\frac{1}{3}\left(\frac{\sqrt{x^2+1000}}{10}\right)^3 - \frac{\sqrt{x^2+100}}{10}\right) + C$

$= \frac{1}{3}(x^2+1000)^{3/2} - 100\sqrt{x^2+100} + C$

(9) $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$

$x = 4\tan\theta$

$dx = 4\sec^2\theta \, d\theta$

$= \displaystyle\int \frac{4\sec^2\theta \, d\theta}{4\sec\theta}$

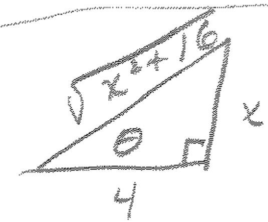
$\sqrt{x^2+16} = \sqrt{16\tan^2\theta + 16}$

$= 4\sec\theta$

$= \displaystyle\int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$\tan\theta = \frac{x}{4} = \frac{opp}{adj}$



$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+16}}{4}$

$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$