§8.3 #31. Find the centroid of the region bounded by the given curves.

$$y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \pi/4$$

$$\overline{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\ dx$$

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}\ dx$$

$$A = \int_a^b (f(x) - g(x))\ dx$$

$$\begin{aligned}
A &= \int_a^b (f(x) - g(x))\ dx = \int_0^{\pi/4} (\cos x - \sin x)\ dx \\
&= [\sin x + \cos x]_0^{\pi/4} = \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - [\sin 0 + \cos 0] \\
&= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - [0 + 1] = \sqrt{2} - 1
\end{aligned}$$

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_a^b x[f(x) - g(x)]\ dx \\
&= \frac{1}{(\sqrt{2} - 1)}\int_0^{\pi/4} x(\cos x - \sin x)\ dx \\
&= \frac{1}{(\sqrt{2} - 1)}\int_0^{\pi/4} \left(\overset{A.}{x\cos x} - \overset{B.}{x\sin x}\right)\ dx
\end{aligned}$$

A. $\int x\cos x\ dx$. Use integration by parts.

$$\begin{aligned}
u &= x & dv &= \cos x\ dx \\
du &= dx & v &= \sin x
\end{aligned}$$

$$\begin{aligned}
\int u\ dv &= uv - \int v\ du \\
&= (x)(\sin x) - \int \sin x\ dx \\
&= x\sin x + \cos x + C
\end{aligned}$$

B. $\int x\sin x\ dx$. Use integration by parts.

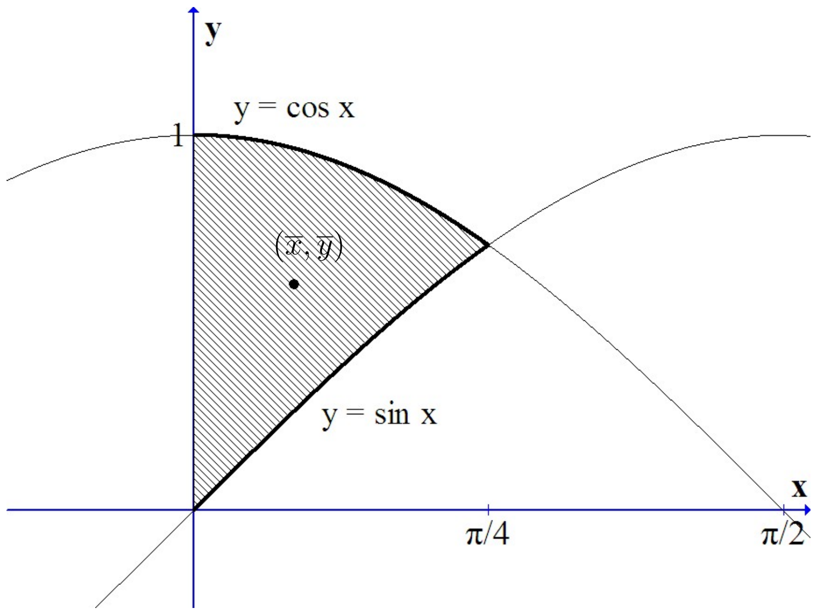
$$\begin{aligned}
u &= x & dv &= \sin x\ dx \\
du &= dx & v &= -\cos x
\end{aligned}$$

$$\begin{aligned}\int u\ dv &= uv - \int v\ du \\ &= (x)(-\cos x) - \int (-\cos x)\ dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

$$\begin{aligned}\overline{x} &= \frac{1}{(\sqrt{2}-1)} \int_0^{\pi/4} (x\cos x - x\sin x)\ dx \\ &= \frac{1}{(\sqrt{2}-1)} \left[(x\sin x + \cos x) - (-x\cos x + \sin x)\right]_0^{\pi/4} \\ &= \frac{1}{(\sqrt{2}-1)} \left[\frac{\pi}{4}\sin\frac{\pi}{4} + \cos\frac{\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right] \\ &\quad - \left[((0)\sin(0) + \cos(0) + (0)\cos(0) - \sin(0)\right] \\ &= \frac{1}{(\sqrt{2}-1)} \left[\frac{\pi}{4}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right] - [0+1+0-0] \\ &= \frac{\frac{\pi\sqrt{2}}{4} - 1}{(\sqrt{2}-1)} = \frac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)} \\ \overline{x} &= \frac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)}\end{aligned}$$

$$\begin{aligned}\overline{y} &= \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]\ dx \\ &= \frac{1}{2(\sqrt{2}-1)} \int_0^{\pi/4} \left[(\cos x)^2 - (\sin x)^2\right]\ dx \\ &\quad \text{Note: } \cos^2 x - \sin^2 x = \cos 2x \\ &= \frac{1}{2(\sqrt{2}-1)} \int_0^{\pi/4} \cos 2x\ dx \\ &= \frac{1}{2(\sqrt{2}-1)} \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} \\ &= \frac{1}{4(\sqrt{2}-1)} \left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right] \\ &= \frac{1}{4(\sqrt{2}-1)} \left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] \\ &= \frac{1}{4(\sqrt{2}-1)}(1-0) = \frac{1}{4(\sqrt{2}+1)} \\ \overline{y} &= \frac{1}{4(\sqrt{2}-1)}\end{aligned}$$

The centroid of the region is $\left(\dfrac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)}, \dfrac{1}{4(\sqrt{2}-1)}\right)$


y
y = cos x
1
$(\overline{x}, \overline{y})$
y = sin x
x
π/4
π/2