§4.4 L'Hospital's Rule

HW §4.4 #5-64

Thurs. Quiz
§4.1-4.3

L'Hospital's Rule

Suppose f and g are differentiable at a and ~~$f'(x)\neq 0$~~ $g'(x)\neq 0$ near $x=a$. ~~Then~~ Also suppose that

$\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$

or that

$\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$

Then $$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example Find the limit.

① $$\lim_{x\to 0} \frac{\sin x}{x} \qquad \sin x \to \sin(0) = 0, \quad x \to 0 \qquad \frac{0}{0} \text{ form}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1$$

② $$\lim_{x\to 1} \frac{\ln x}{x-1} \qquad \ln x \to \ln(1) = 0, \quad x-1 \to 1-1=0 \qquad \frac{0}{0} \text{ form}$$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{\left(\frac{1}{x}\right)}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{(1)} = 1$$

Answer.

③ $$\lim_{x\to\infty} \frac{e^x}{x^2} \qquad e^x \to \infty, \quad x^2 \to \infty \qquad \frac{\infty}{\infty} \text{ form}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \qquad e^x \to \infty, \quad 2x \to \infty \qquad \frac{\infty}{\infty} \text{ form}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2} \qquad e^x \to \infty \qquad = \boxed{\infty}$$

Answer.

④ $\lim_{x \to 0} \frac{e^{(3x)} - e^{7x}}{x} \rightarrow \frac{e^{3\cdot 0} - e^{7\cdot 0} = 1 - 1 = 0}{\rightarrow 0}$ $\frac{0}{0}$ form

$\overset{L'H}{=} \lim_{x \to 0} \frac{3e^{3x} - 7e^{7x}}{1}$

$= 3e^0 - 7e^0 = 3 - 7 = -4$ ✓

⑤ $\lim_{t \to 0} \frac{e^{3t} - 1}{t} \rightarrow \frac{e^0 - 1 = 1 - 1 = 0}{\rightarrow 0}$ $\frac{0}{0}$ form

$\overset{L'H}{=} \lim_{t \to 0} \frac{3e^{3t}}{1} = 3e^0 = 3$

⑥ Tricky Problem: Example 5 p. 301

Find $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$ $\rightarrow \sin \pi = 0$, $\rightarrow 1 - \cos(\pi) = 1 - (-1) = 2$, $\frac{0}{2} = 0$

$= 0/2 = 0$

WRONG: $\overset{L'H}{=} \lim_{x \to \pi^-} \frac{\cos x}{0 - (-\sin x)} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x}$

$= \lim_{x \to \pi^-} \cot x = -\infty$

$\boxed{\text{Answer: } 0}$

§4.4 Continued

Intermediate Products.

If we have $\lim_{x \to a} f(x) = \pm\infty$

and $\lim_{x \to a} g(x) = 0$,

then $\lim_{x \to a} f(x)g(x)$ is

called an indeterminate product $\infty \cdot 0$.

We rewrite this as

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

This is a $\frac{0}{0}$ form.

Note: $\lim_{x \to a} \frac{1}{f(x)} = 0$

OR

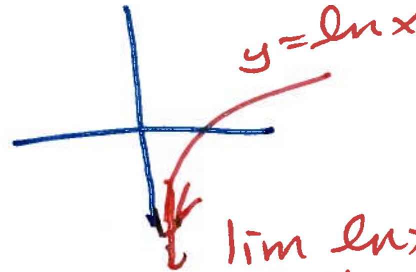
$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} \frac{f(x) \nearrow \infty}{(1/g(x)) \to \infty}$$

This ~~is~~ is an $\frac{\infty}{\infty}$ form

Example: Find the limit.

① $\lim\limits_{x\to 0^+} \overset{\nearrow 0}{x}\, \overset{\nearrow -\infty}{\ln x}$

$= \lim\limits_{x\to 0^+} \dfrac{\ln x \;\nearrow -\infty}{\left(\frac{1}{x}\right) \to \infty}$



$\lim\limits_{x\to 0^+} \ln x = -\infty$

Now use L'Hospital's Rule

$\overset{L'H}{=} \lim\limits_{x\to 0^+} \dfrac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}$

Aside

$$\frac{1}{\left(\frac{1}{3}\right)} = 3$$

$= \lim\limits_{x\to 0^+} \left(\frac{1}{x}\right)\left(-\frac{x^2}{1}\right)$

$= \lim\limits_{x\to 0^+} -x = 0$

Answer

Aside

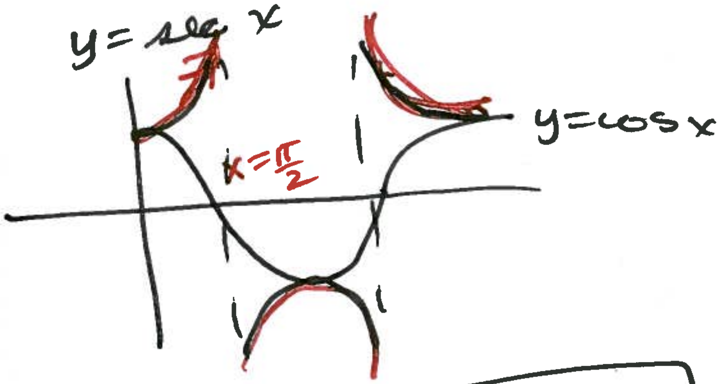
$\left(\frac{1}{x}\right)' = (x^{-1})'$

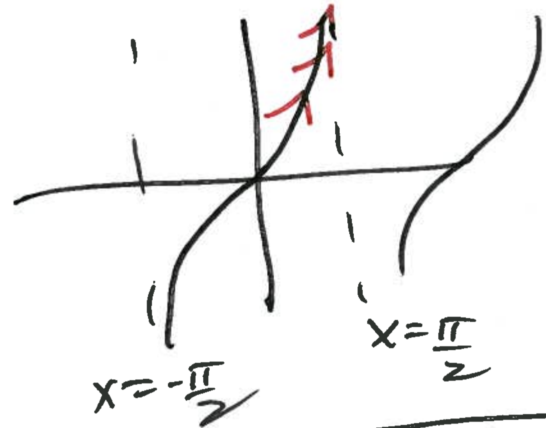
$= -1 \cdot x^{-2}$

$= -\dfrac{1}{x^2}$

Indeterminate Differences

Example: $\displaystyle \lim_{x \to (\pi/2)^-} (\overset{\infty}{\sec x} - \overset{\infty}{\tan x})$





$\boxed{\text{This gives an indeterminate difference } \infty - \infty}$

Write as a fraction

$$\lim_{x \to (\frac{\pi}{2})^-} (\sec x - \tan x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin x}{\cos x}$$

$1 - \sin\frac{\pi}{2} = 1 - 1 = 0$

$\cos(\frac{\pi}{2}) \to 0$

$\frac{0}{0}$ form.

Now use L'Hospital's Rule

$$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{0 - \cos x}{-\sin x} = \frac{+\cos(\pi/2)}{+\sin(\frac{\pi}{2})} = \frac{0}{1} = 0.$$

Indeterminate Powers

We study limits of the form

$$\lim_{x \to a} [f(x)]^{g(x)}$$

There are three indeterminate forms.

type 0^0

type ∞^0

type 1^∞

EXAMPLE Calculate

$$\lim_{x\to 0}(1+x)^{1/x}$$

Aside

$$e^{\ln 5} = 5$$

SOLUTION:

$$\lim_{x\to 0}(1+x)^{x} = \lim_{x\to 0} e^{\boxed{\ln(1+x)^{1/x}}}$$

$$\lim_{x\to 0} \ln(1+x)^{\cancel{1/x}}$$

$$= \lim_{x\to 0} \overset{\infty}{\frac{1}{x}} \ln(\overset{\ln 1 = 0}{1+x})$$

Make this into a fraction

$$\ln(1+0) = \ln 1 = 0$$

$$= \lim_{x\to 0} \frac{\ln(1+x)}{x \to 0} \qquad \frac{0}{0} \text{ form}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\left(\frac{1}{1+x}\right)}{1} = \frac{1}{1+0} = 1$$

Answer: $e^1 = e$

So $\boxed{e = \lim_{x\to 0}(1+x)^{1/x}}$

Example $\displaystyle \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$ $\quad 1^{\infty}$ form

SOLUTION

$$= \lim_{x\to\infty} e^{\boxed{\ln\left(1+\frac{3}{x}\right)^{2x}}}$$

$$\lim_{x\to\infty} \ln\left(1+\frac{3}{x}\right)^{2x}$$

$$= \lim_{x\to\infty} 2x \ln\left(1+\frac{3}{x}\right)$$

$$= \lim_{x\to\infty} \frac{2\ln\left(1+\frac{3}{x}\right)}{\left(\frac{1}{x}\right)}$$

Aside

- $\left(\frac{1}{x}\right)' = \left(x^{-1}\right)'$
  $= -1 \cdot x^{-2}$
  $= \frac{-1}{x^2}$
- $\left(\frac{3}{x}\right)' = 3\left(\frac{1}{x}\right)'$
  $= 3\left(\frac{-1}{x^2}\right)$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2\left(\frac{1}{(1+3/x)}\right)\cdot\left(3\cancel{\left(\frac{-1}{x^2}\right)}\right)}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} 2\left(\frac{1}{1+3/x}\right)\cdot 3 = 2\cdot 3 = 6$$

$\to 0$

Final answer: $\boxed{e^6}$

§4.4 #13 $\displaystyle\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)}$ $\begin{array}{l}\to \tan 0 = 0 \\ \to \tan 0 = 0\end{array}$ $\frac{0}{0}$ form

$$\overset{L'H}{=} \lim_{x\to 0} \frac{p\sec^2(px)}{q\sec^2(qx)} = \frac{p\sec^2(p\cdot 0)}{q\sec^2(q\cdot 0)}$$

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

$$= \boxed{\frac{p}{q}} \leftarrow \text{answer}$$

§4.1 #41 Find the ~~absolute max & min~~ critical numbers.

$$f(\theta) = 2\cos\theta + \sin^2\theta$$

SOLUTION:

$$f(\theta) = 2\cos\theta + (\sin\theta)^2$$

$$f'(\theta) = -2\sin\theta + 2(\sin\theta)\cdot\cos\theta$$

$$= 2\sin\theta(-1 + \cos\theta)$$

Solve $f'(\theta) = 0$

$$2\sin\theta(-1 + \cos\theta) = 0$$

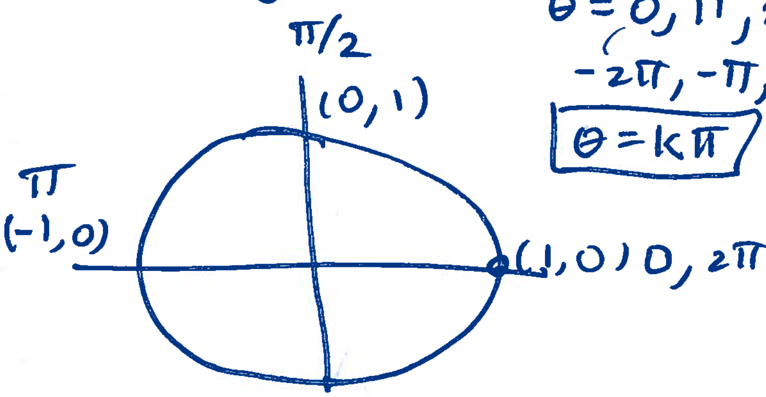
$\sin^{-1}(0) = \theta$
$\theta = 0$

$\sin\theta = 0$ or $-1 + \cos\theta = 0$

$\theta = 0, \pi, 2\pi, 3\pi$ $\quad$ $\cos\theta = 1$

$-2\pi, -\pi, \ldots$ $\quad$ $\boxed{\theta = \ldots -2\pi, 0, 2\pi, 4\pi \ldots}$

$\boxed{\theta = k\pi}$



Answer: $\theta = k\pi$

$= \{\ldots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi \ldots\}$

$= \{k\pi \mid k \text{ an integer}\}$