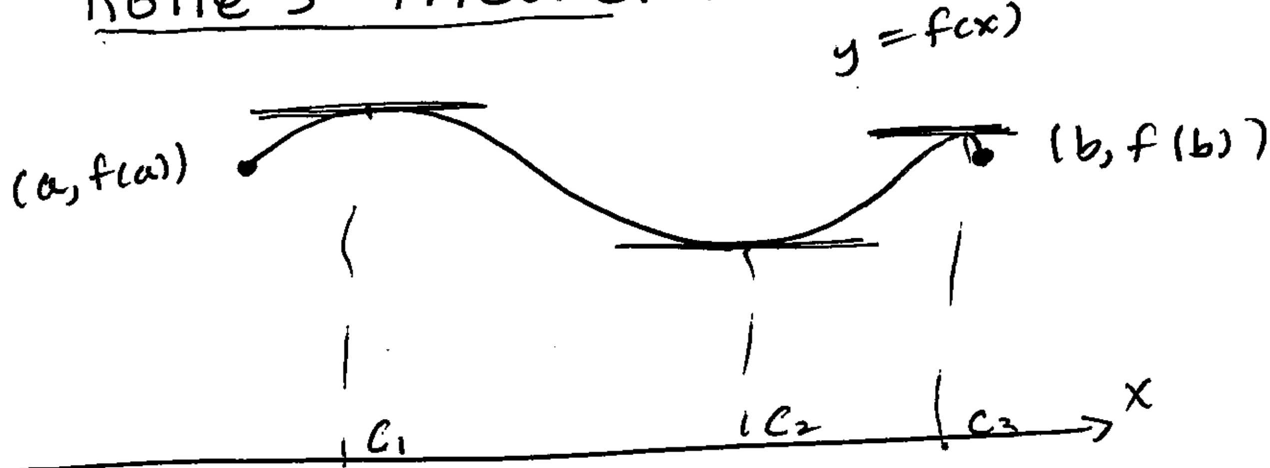
§4.2 The Mean Value Theorem

§4.2 HW # 1-4, 11-14

Rolle's Theorem



Rolle's Theorem

Assume

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

EXAMPLE Verify that the function satisfies the three hypotheses of Rolle's Theorem. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = x^3 - x^2 - 6x + 2, \quad [0, 3].$$

SOLUTION

- f is contin on $[0, 3]$
- f is ~~de~~ differentiable on $(0, 3)$ because it is a polynomial
- $f(0) = 2$

$f(3) = (3)^3 - (3)^2 - 6(3) + 2$

$= 27 - 9 - 18 + 2 = 2$

So $f(0) = f(3)$.

Next, find c in $(0, 3)$ such that $f'(c) = c$

$f'(x) = 3x^2 - 2x - 6$

solve $f'(x) = 0$

$3x^2 - 2x - 6 = 0 \quad a = 3, b = -2, c = -6$

$\cancel{(3x + 2)(x - 3)}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{2 \pm \sqrt{76}}{6}$$

$76 = 2^2 \cdot 19 = 4 \cdot 19$

76

2 38

2 19

$\sqrt{76} = \sqrt{4 \cdot 19}$

$= 2\sqrt{19}$

$$x = \frac{2 \pm 2\sqrt{19}}{6}$$

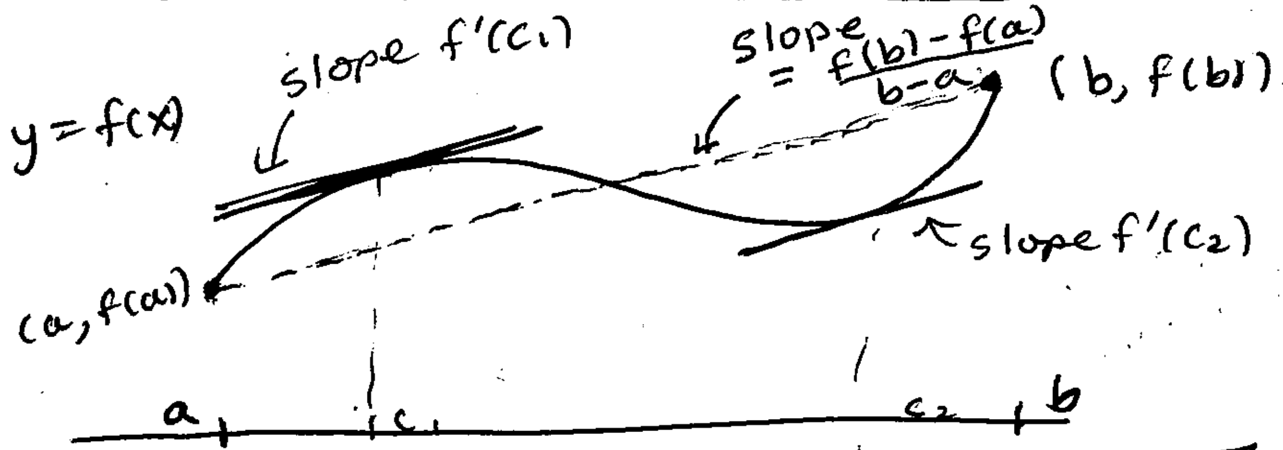
$$x = \frac{\cancel{2}(1 \pm \sqrt{19})}{\cancel{2}(3)}$$

$$x = \frac{1 \pm \sqrt{19}}{3}$$

Only $x = \frac{1+\sqrt{19}}{3}$ is in $(0, 3)$

$$\boxed{c = \frac{1+\sqrt{19}}{3}}$$

The Mean Value Theorem



If
- $f(x)$ is continuous on $[a,b]$
- $f(x)$ is differentiable on (a,b).

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof: Let $L(x)$ be the line through $(a, f(a))$ and $(b, f(b))$.

Note:
- $L'(x) = \frac{f(b)-f(a)}{b-a}$ ← the slope of the line
- $L(a) = f(a)$, $L(b) = f(b)$

Now let $F(x) = f(x) - L(x)$.

$F(a) = f(a) - L(a) = 0$, $F(b) = f(b) - L(b) = 0$

F is cont on $[a,b]$, diff. on (a,b)

By Rolle's Thm there is a number c in (a,b) such that $F'(c)=0$.

But $F'(x) = f'(x) - L'(x)$

so $0 = F'(c) = f'(c) - L'(c)$

$$f'(c) - L'(c) = 0$$

$$f'(c) = L'(c)$$

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad \square$$

EXAMPLE: Verify that the function satisfies the hypothesis of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = x^3 - x, \quad [0, 2].$$

SOLUTION ~~Hy~~ Hypothesis

- $f(x)$ is continuous on $[0,2]$
- $f(x)$ is differentiable $(0,2)$

because it is a polynomial.

conclusion $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

LHS
$$f(x) = x^3 - x$$
$$f'(x) = 3x^2 - 1$$
$$f'(c) = 3c^2 - 1$$

RHS $a=0$, ~~b=a~~ $b=2$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{[(2)^3-(2)] - [0^3-0]}{2}$$
$$= \frac{8-2}{2} = \frac{6}{2} = 3.$$

Set LHS = RHS

$$3c^2 - 1 = 3$$
$$3c^2 = 4$$
$$c^2 = \frac{4}{3}$$

$$c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$

only $\boxed{c = \frac{2\sqrt{3}}{3}}$ is in $(0,2)$.

EXAMPLE Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

SOLUTION: by MVT

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Let $a=0$, $b=2$.

So $$f'(c) = \frac{f(2) - f(0)}{2-0}$$

$$f'(c) = \frac{f(2) - (-3)}{2}$$

but $f'(c) \leq 5$

$$\frac{f(2)+3}{2} = f'(c) \leq 5$$

$$\frac{f(2)+3}{2} \leq 5$$

$$f(2) + 3 \leq 10$$

$$f(2) \leq 7 \qquad \square$$