§3.6 Use logarithmic differentiatio[n] to find the derivative.

$$y = \sqrt{x}^{\,x}$$

$$y = (x^{1/2})^x$$

$$y = x^{x/2}$$

$$\ln y = \ln x^{x/2}$$

$$\ln y = \frac{x}{2} \ln x$$

$$\frac{1}{y} y' = \left(\frac{x}{2}\right)'(\ln x) + \left(\frac{x}{2}\right)(\ln x)'$$

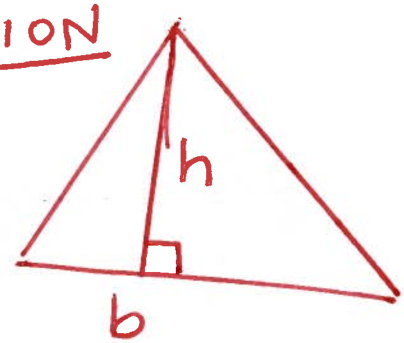
$$\frac{1}{y} y' = \frac{1}{2} \ln x + \left(\frac{x}{2}\right)\left(\frac{1}{x}\right)$$

$$y' = y\left[\frac{1}{2} \ln(x) + \frac{1}{2}\right]$$

$$y' = \sqrt{x}^{\,x}\left[\frac{1}{2} + \frac{1}{2} \ln x\right]$$

§3.9 #19 The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of $2 \text{ cm}^2/\text{min}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

SOLUTION



FORMULA: $A = \frac{1}{2} bh$

Facts: $\frac{dh}{dt} = 1 \text{ cm/min}$

$\frac{dA}{dt} = 2 \text{ cm}^2/\text{min}$

altitude means height, h

Find: $\frac{db}{dt}$ when $h = 10$, $A = 100$.

Derivative:

$A = \frac{1}{2} bh$

$A' = \frac{1}{2}(b'h + bh')$

FILL-IN $h = 10, \ A = 100, \ b = ?$

$A' = 2, \quad h' = 1$

Snapshot to find b.

$$A = \frac{1}{2} bh$$

$$100 = \frac{1}{2} b(10)$$

$$100 = 5b, \qquad b = 20$$

$$A' = \frac{1}{2}(b'h + bh')$$

$$2 = \frac{1}{2}(b' \cdot 10 + 20 \cdot 1)$$

$$2 = 5b' + 10$$

$$-8 = 5b' \qquad b' = \frac{-8}{5} \ \text{cm/min}$$

§ 3.10 #35 The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error? The percentage error?

$A = 4\pi r^2$, $C = 2\pi r$

Differential FORMULA

$dy = \left(\frac{dy}{dx}\right) dx$ ← error in x

dy: approx. error in y

$\frac{dy}{dx}$: y' or $f'(x)$ derivative

$r = \frac{C}{2\pi}$

$A = 4\pi\left(\frac{C}{2\pi}\right)^2 = \frac{4\pi C^2}{4\pi^2}$

$A = \frac{C^2}{\pi}$

$$dA = A' \, dC$$

$$dA = \frac{2C}{\pi} \, dC$$

put $C = 84$, $dC = .5$

$$dA = \frac{2}{\pi}(84)(.5)$$

$$\boxed{dA = \frac{84}{\pi} \approx 26.7 \text{ cm}^2}$$

Relative Error $\quad \dfrac{dA}{A} = \dfrac{\left(\frac{2C\,dC}{\pi}\right)}{\left(\frac{C^2}{\pi}\right)}$

$$= \frac{2C\,dC}{\pi} \cdot \frac{\pi}{C^2} = 2\frac{dC}{C}$$

$dC = .5$, $C = 84$

$$\frac{dA}{A} = \frac{2(.5)}{84} = \frac{1}{84} \approx 0.0119$$

Percentage Error $\quad 1.19\%$

Practice Test #7

Find the equation of the tangent line at $(-1, 2)$ to the curve given by

$$x^2 + xy^2 + y^3 = 5$$

SOLUTION

Find the slope.

$$2x + [(x)'(y^2) + (x)(y^2)'] + 3y^2y' = 0$$

$$2x + [1 \cdot y^2 + x2y \cdot y'] + 3y^2y' = 0$$

put in $x = -1$, $y = 2$, $m = y'$

$$2(-1) + (2)^2 + 2(-1)(2)m + 3(2)^2 m = 0$$

$$-2 + 4 - 4m + 12m = 0$$

$$2 + 8m = 0$$

$$8m = -2$$

$$m = -2/8 = -\frac{1}{4}$$

$m = -\frac{1}{4}$, $x_1 = -1$, $y_1 = 2$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{1}{4}(x - (-1))$$

$$y - 2 = -\frac{1}{4}(x + 1)$$

$$y - 2 = -\frac{1}{4}x - \frac{1}{4}$$

$$y = -\frac{1}{4}x - \frac{1}{4} + 2$$

$$y = -\frac{1}{4}x - \frac{1}{4} + \frac{8}{4}$$

$$\boxed{y = -\frac{1}{4}x + \frac{7}{4}}$$

Practice Test 6a)

Use implicit differentiation to find dy/dx.

SOLUTION $$xy^4 + x^2y = x + 3y$$

$$[(x)'(y^4) + (x)(y^4)'] + [(x^2)'(y) + (x^2)(y)'] = 1 + 3y'$$

$$[1 \cdot y^4 + x \cdot 4y^3y'] + [2xy + x^2y'] = 1 + 3y'$$

$$y^4 + 4xy^3y' + 2xy + x^2y' = 1 + 3y'$$

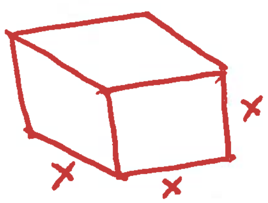
$$4xy^3y' + x^2y' - 3y' = 1 - y^4 - 2xy$$

$$y'[4xy^3 + x^2 - 3] = 1 - y^4 - 2xy$$

$$y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$$

§3.1D #33 a) The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing the volume of the cube.

SOLUTION



Differential formula

$$dy = \left(\frac{dy}{dx}\right)dx$$

$$dy = (y')\,dx$$

$$dy = (f'(x))\,dx$$

$$V = x^3$$

$$dV = \left(\frac{dV}{dx}\right)dx \qquad dV = V'(x)\,dx$$

$$dV = (3x^2)\,dx$$

put in $x=30$, $dx=.1$

$$dV = (3(30)^2)(.1)$$

$$\boxed{dV = 270\text{ cm}^3}$$

Relative error: $\dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = \dfrac{3\,dx}{x}$

put in $x=30$, $dx=.1$

$$= \frac{3(.1)}{30} = \frac{.1}{10} = \boxed{.01}$$

Percentage error: 1%

§3.10 #14 Find the differential of each function.

(a) $y = e^{\tan \pi t}$

$$dy = \left(\frac{dy}{dx}\right) dx$$

$$dy = \left(e^{\tan \pi t}(\sec^2 \pi t)(\pi)\right) dx$$

$\frac{dy}{dx}$

$$dy = \left(\pi \sec^2(\pi t)\, e^{\tan \pi t}\right) dx$$

b) $y = \sqrt{1 + \ln x}$

$$y = (1 + \ln x)^{1/2}$$

$$dy = \left(\frac{dy}{dx}\right) dx$$

$$dy = \left[\frac{1}{2}(1 + \ln x)^{-1/2} \cdot \frac{1}{x}\right] dx$$

$$dy = \frac{1}{2x\sqrt{1 + \ln x}}\, dx$$

§3.6 #39 Use logarithmic differentiation to find the derivative.

$$y = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}$$

SOLUTION

$$\ln y = \ln\left(\frac{\sin^2 x \tan^4 x}{(x^2+1)^2}\right)$$

$$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2$$

$$\ln y = 2\ln(\sin x) + 4\ln(\tan x) - 2\ln(x^2+1)$$

$$\frac{1}{y}y' = 2\frac{1}{\sin x}\cdot\cos x + 4\frac{1}{\tan x}\cdot\sec^2 x - 2\left(\frac{1}{x^2+1}\right)2x$$

$$y' = y\left[2\cot x + 4\frac{\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right]$$

$$y' = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left[2\cot x + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right]$$

§3.6 #46 Use logarithmic differentiation to find the derivative of the function.

$$y = (\sin x)^{\ln x}$$

SOLUTION

$$\ln y = \ln (\sin x)^{\ln x}$$

$$\ln y = (\ln x) \cdot \ln(\sin x)$$

$$\frac{1}{y} y' = \overbrace{(\ln x)' \cdot (\ln(\sin x)) + (\ln x)(\ln(\sin x))'}^{\text{product rule}}$$

$$\frac{1}{y} y' = \left(\frac{1}{x}\right)(\ln(\sin x)) + (\ln x) \cdot \left(\frac{1}{\sin x} \cdot \cos x\right) \text{ chain rule}$$

$$y' = y\left[\frac{\ln(\sin x)}{x} + (\ln x)(\cot x)\right]$$

$$y' = (\sin x)^{\ln x}\left[\frac{\ln(\sin x)}{x} + (\ln x)(\cot x)\right]$$

§3.6 # 45 Use logarithmic Differentiat. to find $\frac{dy}{dx}$.

$$y = (\cos x)^x$$

SOLUTION

$$\ln y = \ln(\cos x)^x$$

$$\ln y = x \ln(\cos x)$$

$$\frac{1}{y} y' = (x)'(\ln(\cos x)) + (x)(\ln(\cos x))'$$

$$\frac{1}{y} y' = 1 \cdot \ln(\cos x) + x\left(\frac{1}{\cos x} \cdot -\sin x\right)$$

$$y' = y[\ln(\cos x) - x \tan x]$$

$$y' = (\cos x)^x[\ln(\cos x) - x \tan x]$$

§3.3 #41

Assume $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$

$$\lim_{t \to 0} \frac{\tan 6t}{\sin 2t}$$

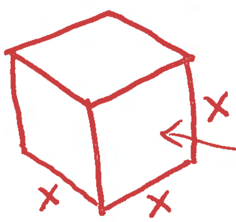
$$= \lim_{t \to 0} (\tan 6t)\left(\frac{1}{\sin 2t}\right)$$

$$= \lim_{t \to 0} \left(\frac{\sin 6t}{\cos 6t}\right)\left(\frac{1}{\sin 2t}\right)$$

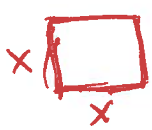
$$= \lim_{t \to 0} (\sin 6t) \underbrace{\frac{1}{\cos(6t)}}_{\cos(0)=1} \frac{1}{\sin(2t)}$$

$$= \lim_{t \to 0} \underbrace{\left(\frac{\sin(6t)}{(6t)}\right)}_{1} \cdot \underbrace{\left(\frac{(2t)}{\sin(2t)}\right)}_{1} \cdot \frac{6}{2} = 3$$

§3.10 #33b The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing the surface area of the cube.



The area of one face is x^2.



There are six faces.

$$A = 6x^2$$

FORMULA FOR DIFFERENTIAL

$$dy = \left(\frac{dy}{dx}\right) dx$$

$$dA = \left(\frac{dA}{dx}\right) dx$$

$$dA = (12x)\,dx$$

put in $x = 30$, $dx = .1$

$$dA = (12)(30)(.1)$$

$$\boxed{dA = 36 \text{ cm}^2}$$

relative error $\frac{dA}{A} = \frac{12x\,dx}{6x^2}$

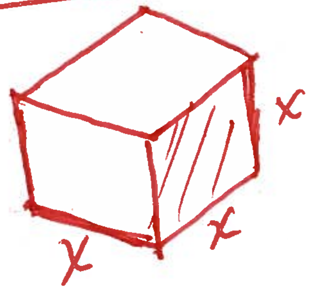
$= \frac{2\,dx}{x}$ put in $dx = .1$, $x = 30$

$= \frac{2(.1)}{30} = 0.00\overline{6}$

percentage error $.\overline{6}\% \approx \boxed{.67\%}$

$= \frac{2}{3}\%$

A Cube

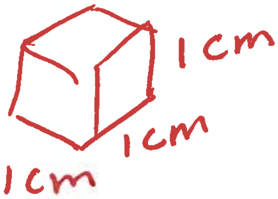


Volume: $V = x^3$

Area: $A = 6x^2$

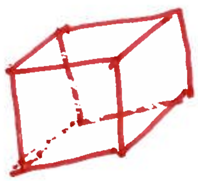
six faces

area of one face



$V = 1\ cm^3$

$= 1cc$



Circle:

$A = \pi r^2$

$C = 2\pi r$

Practice Test 6b)

Use implicit differentiation to find $\frac{dy}{dx}$.

$$\sin(xy) = x^2 - y^2$$

SOLUTION:

$$\overbrace{\cos(xy)\cdot(xy)'}^{\text{chain rule}} = 2x - 2yy'$$

$$\cos(xy)\cdot\overbrace{[x'y + xy']}^{\text{product rule}} = 2x - 2yy'$$

$$\cos(xy)\,[1\cdot y + xy'] = 2x - 2yy'$$

$$y\cos(xy) + y'\cdot x\cos(xy) = 2x - 2yy'$$

$$y'\cdot x\cos(xy) + 2yy' = 2x - y\cos(xy)$$

$$y'\,[x\cos(xy) + 2y] = 2x - y\cos(xy)$$

$$y' = \frac{2x - y\cos(xy)}{x\cos(xy) + 2y}$$

§3.11 #39

$$y = \arctan(\tanh x)$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

Chain Rule

$$y' = \frac{1}{1+(\tanh x)^2} \cdot \operatorname{sech}^2 x$$

$\frac{d}{dx}$ (inside $\tanh x$)

$$y' = \frac{\operatorname{sech}^2 x}{1+\tanh^2 x}$$

§ 3.5 # 27

$x^2+y^2=(2x^2+2y^2-x)^2$

Find the tangent line at the point $(0, 1/2)$.

SOLUTION: Find the slope

$2x+2yy'=2(2x^2+2y^2-x)^1\cdot(4x+4yy'-1)$

Put in $x=0$, $y=1/2$, $m=y'$

$2(0)+2(\frac{1}{2})m=2\left(2(0)^2+2(\frac{1}{2})^2-(0)\right)\cdot\left(4(0)+4(\frac{1}{2})m-1\right)$

$m=2(\frac{1}{2})(2m-1)$

$m=2m-1$

$-m=-1$

$m=1$

$x_1=0,\ y_1=1/2$

$y-y_1=m(x-x_1)$

$y-1/2=1\cdot(x-0)$

$\boxed{y=x+1/2}$

$\S 3.2$ #41 $\quad f(x) = y = \dfrac{x^2}{1+x}$

Find $\quad f''(1)$

$$f'(x) = \frac{(x^2)'(1+x) - (x^2)(1+x)'}{(1+x)^2}$$

$$= \frac{2x(1+x) - (x^2)(1)}{(1+x)^2}$$

$$= \frac{2x + 2x^2 - x^2}{(1+x)^2}$$

$$= \frac{x^2+2x}{(x+1)^2}$$

$$f''(x) = \frac{(x^2+2x)'(x+1)^2 - (x^2+2x)\left[(x+1)^2\right]'}{\left[(x+1)^2\right]^2}$$

$$f''(x) = \frac{(2x+2)(x+1)^2 - (x^2+2x)2(x+1)}{(x+1)^4}$$

We can put in $x=1$ here. It's easier.

$$= \frac{2(x+1)\cdot(x+1)^2 - (x^2+2x)2\cdot(x+1)}{(x+1)(x+1)^3}$$

$$= \frac{\cancel{(x+1)}\left[2(x+1)^2 - 2(x^2+2x)\right]}{\cancel{(x+1)}\cdot(x+1)^3}$$

$$= \frac{\left[2(x^2+2x+1) - 2x^2 - 4x\right]}{(x+1)^3}$$

$$= \frac{\cancel{2x^2} + \cancel{4x} + 2 - \cancel{2x^2} - \cancel{4x}}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3}$$

$$f''(1) = \frac{2}{(1+1)^3} = \frac{2}{2^3} = \frac{1}{4}$$

Practice Test

9(a) find the derivative.

$y = \arctan \sqrt{x}$

$y' = \left(\frac{1}{1+(\sqrt{x})^2}\right) \cdot \frac{d}{dx}\sqrt{x}$

$y' = \left(\frac{1}{1+x}\right)\frac{1}{2\sqrt{x}}$

$$\boxed{\begin{array}{l}\text{FORMULA} \\ \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\end{array}}$$

$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2}$

$= \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Practice Test

9b) Find the derivative

$$y = x \arcsin(x) + \sqrt{1-x^2}$$

$$y' = (x)'(\arcsin(x)) + (x)(\arcsin(x))' + \left[(1-x^2)^{1/2}\right]'$$

FORMULA

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$y' = 1 \cdot \arcsin(x) + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$$

$$y' = \arcsin x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$\boxed{y' = \arcsin x}$$

Practice Test #7 (more like this on §3.5 # 25–30)

Find the equation of the tangent line to $x^2 + xy^2 + y^3 = 5$ at the point $(-1, 2)$.

SOLUTION Find the slope.

$$2x + \left[(x)'(y^2) + (x)(y^2)'\right] + 3y^2y' = 0$$

$$2x + 1 \cdot y^2 + x \cdot 2yy' + 3y^2y' = 0$$

Put in $x = -1$, $y = 2$, $m = y'$

$$2(-1) + (2)^2 + (-1)(2)(2)m + 3(2)^2m = 0$$

$$-2 + 4 - 4m + 12m = 0$$

$$2 + 8m = 0$$

$$8m = -2, \quad m = -\frac{1}{4}$$

$x_1 = -1$, $y_1 = 2$, $m = -\frac{1}{4}$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{1}{4}(x - (-1))$$

$$y - 2 = -\frac{1}{4}x - \frac{1}{4}$$

$$y = -\frac{1}{4}x - \frac{1}{4} + 2, \quad y = -\frac{1}{4}x - \frac{1}{4} + \frac{8}{4}$$

$$\boxed{y = -\frac{1}{4}x + \frac{7}{4}}$$