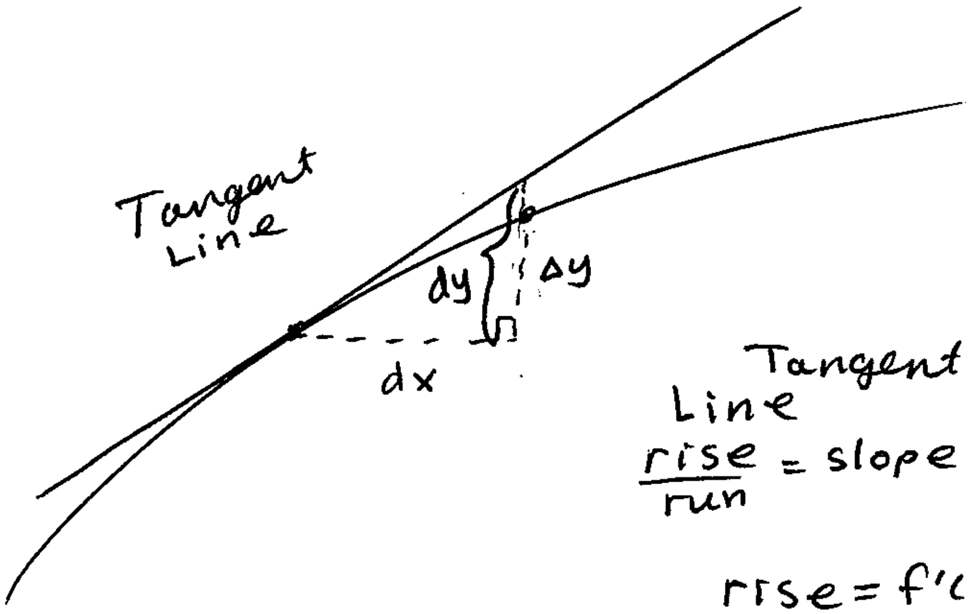
relative error $\frac{dV}{V} = \frac{179}{\left(\frac{(84)^3}{6\pi^2}\right)} = \frac{(179)\cdot 6\pi^2}{(84)^3}$

$c = 84$

≈ 0.018



Tangent Line

$$\frac{\text{rise}}{\text{run}} = \text{slope} = \frac{dy}{dx} = f'(x)$$

$$\text{rise} = f'(x)\,\text{run}$$

$$dy = f'(x)\,dx$$

<u>Example</u>: $f(x) = \sqrt{x} = x^{1/2}$.

We measure $x = 4$, with a possible error of 0.1. Use differentials to estimate the error in $f(x) = \sqrt{x}$.

<u>SOLUTION</u>: $f'(x) = \frac{1}{2}x^{-1/2}$

$$f'(4) = \frac{1}{2}(4)^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{4}}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}$$

$$dy = f'(x)\,dx$$

$$dy = \frac{1}{4}dx$$

$$dy = \frac{1}{4}(.1) = \frac{.1}{4} = \left(\frac{.1}{4}\right)\left(\frac{25}{25}\right) = \frac{2.5}{100}$$

$$= \frac{25}{1000} = \boxed{.025}$$

§3.10 #34 The radius of a circular disk is given as 24cm with a maximum error in measurement of 0.2cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

SOLUTION $A = \pi r^2$

FORMULA FOR DIFFERENTIAL

$dA = A' \, dr$ ← We saw $dy = f'(x)\,dx$

small change in r.

$dr = .2$

$A' = 2\pi r, \quad r = 24$

$A'(24) = 2\pi(24) = 48\pi$

$dA = A'(24)\,dr$

$dA = 48\pi(.2)$

$dA = 9.6\pi \approx 30.2 \text{ cm}^2$

b) What is the relative error? What is the percentage error?

• relative error: $\dfrac{dA}{A} = \dfrac{dA}{\pi r^2} = \dfrac{30.2}{\pi(24)^2} \approx 0.01\overline{6}$

$r = 24$

• percentage error: $1.\overline{6}\% \approx 1.67\%$

$= 1\tfrac{2}{3}\%$

§3.6 #47 Use logarithmic differentiation to find the derivative of the function.

$$y = (\tan x)^{1/x}$$

$$\ln y = \ln (\tan x)^{1/x}$$

$$\ln y = \frac{1}{x} \ln(\tan x)$$

$$\ln y = (x^{-1})(\ln(\tan x))$$

$$\frac{1}{y} y' = (x^{-1})'(\ln(\tan x)) + (x^{-1})(\ln(\tan x))'$$

$$\frac{1}{y} y' = (-1 \cdot x^{-2})(\ln(\tan x)) + (x^{-1})\left(\frac{1}{\tan x}\right)\sec^2 x$$

$$\frac{1}{y} y' = \frac{-\ln(\tan x)}{x^2} + \frac{\sec^2 x}{x \tan x}$$

$$y' = y\left(-\frac{\ln(\tan x)}{x^2} + \frac{\sec^2 x}{x \tan x}\right)$$

$$y' = (\tan x)^{1/x}\left(-\frac{\ln(\tan x)}{x^2} + \frac{\sec^2 x}{x \tan x}\right)$$

§3.9 #5 A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing.

SOLUTION

FORMULA: $V = \pi r^2 h$

$V = \pi (5)^2 h$

$V = 25\pi h$

Fact: $\frac{dV}{dt} = 3$ m^3/min

Find: $\frac{dh}{dt}$

Derivative $V = 25\pi h$

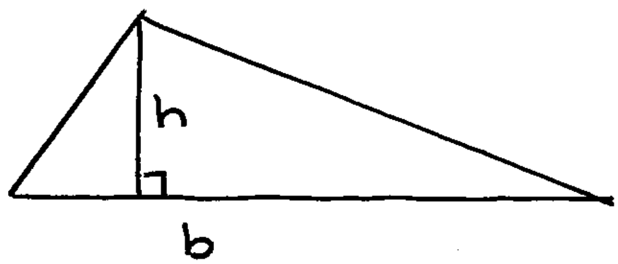
$V' = 25\pi h'$

FILL-IN: $3 = 25\pi h'$

$h' = \frac{3}{25\pi}$

$\frac{dh}{dt} = \frac{3}{25\pi}$ cm/min.

§ 3.9 # 19 The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm²/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm²?



FACTS: $\frac{dh}{dt} = 1$ cm/min

$\frac{dA}{dt} = 2$ cm²/min

FORMULA: $A = \frac{1}{2}bh$

Find: $\frac{db}{dt}$ when $h = 10$, $A = 100$

DERIVATIVE:

$A = \frac{1}{2}bh$

$A' = \frac{1}{2}(b'h + bh')$

FILL-IN: $h' = 1$, $A' = 2$, $h = 10$, $A = 100$, $b = ?$

Find b'

Snapshot

$A = \frac{1}{2}bh$

$100 = \frac{1}{2}b(10)$

$100 = 5b$

$b = 20$

$A' = \frac{1}{2}(b'h + bh')$

$2 = \frac{1}{2}(b' \cdot 10 + 20 \cdot 1)$

$2 = 5b' + 10$

$-5b' = +8$

$b' = -8/5 = -1.6$ cm/min

§3.5 #17 Find $\frac{dy}{dx}$ using implicit differentiation.

$$\sqrt{xy} = 1 + x^2 y$$

SOLUTION

$$(xy)^{1/2} = 1 + x^2 y$$

Take the deriv. with respect to x.

$$x^{1/2} y^{1/2} = 1 + x^2 y$$

product rule

$$(x^{1/2})'(y^{1/2}) + (x^{1/2})(y^{1/2})' = 0 + [(x^2)'(y) + (x^2)(y)'$$

$$\frac{1}{2} x^{-1/2} y^{1/2} + x^{1/2} \frac{1}{2} y^{-1/2} \cdot y' = 2xy + x^2 y'$$

$$\frac{y^{1/2}}{2x^{1/2}} + \frac{x^{1/2}}{2y^{1/2}} y' = 2xy + x^2 y'$$

$$\frac{x^{1/2}}{2y^{1/2}} y' - x^2 y' = 2xy - \frac{y^{1/2}}{2x^{1/2}}$$

$$y' \left(\frac{x^{1/2}}{2y^{1/2}} - x^2 \right) = 2xy - \frac{y^{1/2}}{2x^{1/2}}$$

$$y' = \left(\frac{2xy - \frac{y^{1/2}}{2x^{1/2}}}{\frac{x^{1/2}}{2y^{1/2}} - x^2} \right) \frac{2x^{1/2} y^{1/2}}{2x^{1/2} y^{1/2}}$$

$$y' = \frac{4xy\, x^{1/2} y^{1/2} - y}{x - 2x^2 x^{1/2} y^{1/2}}$$

$$y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

§3.6 #41 Use logarithmic differentiation to find the derivative of the function.

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = (x)'(\ln x) + (x)(\ln x)'$$

$$\frac{1}{y} y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$y' = y\left(\ln(x) + 1\right)$$

$$y' = x^x(1 + \ln x)$$

Practice Test ch 3

6 b) Find $\frac{dy}{dx}$ using implicit differentiation

$$\sin(xy) = x^2 - y^2$$

SOLUTION: $\cos(xy) \cdot (xy)' = 2x - 2yy'$

$$\cos(xy)(x'y + xy') = 2x - 2yy'$$

$$\cos(xy) \cdot (1 \cdot y + xy') = 2x - 2yy'$$

$$y\cos(xy) + y' \cdot x\cos(xy) = 2x - 2yy'$$

$$y' x\cos(xy) + 2yy' = 2x - y\cos(xy)$$

$$y'(x\cos(xy) + 2y) = 2x - y\cos(xy)$$

$$y' = \frac{2x - y\cos(xy)}{2y + x\cos(xy)}$$

Chapter 3 Practice Test

§3.10

15(a) Find the differential dy and (b) evaluate dy given values of x and dx.

$y = \frac{1}{x+1}, \quad x = 1, \quad dx = -.01$

SOLUTION: FORMULA $dy = f'(x)\,dx$

$$f(x) = \frac{1}{x+1}$$

$$f'(x) = \frac{(1)'(x+1) - (1)(x+1)'}{(x+1)^2}$$

$$= \frac{0(x+1) - (1)(1)}{(x+1)^2} = \frac{-1}{(x+1)^2}$$

$$\boxed{a) \quad dy = \frac{-1}{(x+1)^2}\,dx}$$

b) $x = 1, \quad dx = -.01$

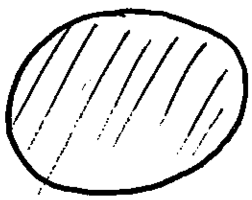
$$dy = \frac{-1}{(1+1)^2}(-.01) = \frac{.01}{2^2} = \frac{.01}{4}$$

$$\boxed{dy = 0.0025}$$

§3.10 #34 The radius of a circular disk is given as 24 cm with a maximum error of 0.2 cm in measurement.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

SOLUTION



$A = \pi r^2$

FORMULA: $dA = A'(r)\,dr$

$A' = 2\pi r$, $r = 24$, $dr = 0.2$

$A'(24) = 2\pi(24) = 48\pi$

$dA = A'(24)\,dr = (48\pi)(.2)$

$dA = 9.6\pi \approx 30.2 \text{ cm}^2$

(b) What is the relative error?

$$\frac{dA}{A} = \frac{dA}{A} = \frac{9.6\pi}{576\pi} \approx 0.0167$$

$A(24) = \pi(24)^2 = 576\pi$

(c) What is the percentage error 1.67%

$1\frac{2}{3}\%$

§3.11 #39 Find the derivative. Simplify where possible.

$y = \arctan(\tanh x)$

FORMULAS

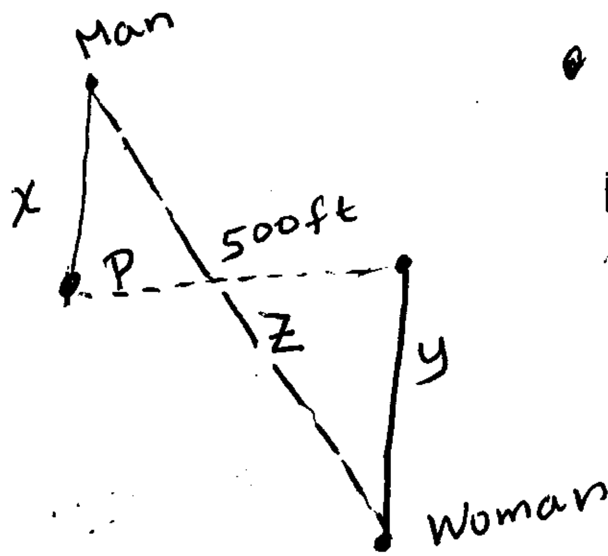
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$
- $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$$y' = \frac{1}{1+(\tanh x)^2} \cdot \operatorname{sech}^2 x$$

$$\boxed{y' = \frac{\operatorname{sech}^2 x}{1+(\tanh x)^2}}$$

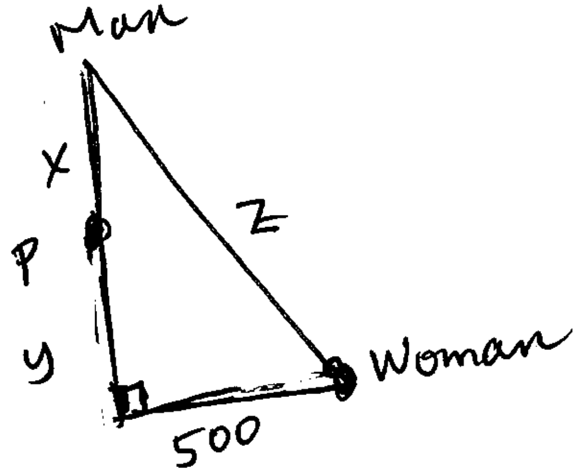
§3.9 #17 A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

SOLUTION



• Find $\frac{dz}{dt}$

15 min. after woman starts.



FACTS: $\frac{dx}{dt} = 4 \text{ ft/s}$

$\frac{dy}{dt} = 5 \text{ ft/s}$

FORMULA:

$(x+y)^2 + (500)^2 = z^2$

DERIVATIVE

$\cancel{2}(x+y)\cdot(x'+y') = \cancel{2}zz'$

FILL-IN: At 15 min after woman begins.

velocity $= \frac{\text{dist}}{\text{time}}$

dist = velocity × time

Man $x = (4 \text{ ft/s})(20 \text{ min}) = (4 \text{ ft/}\cancel{s})\left(\frac{60\cancel{s}}{1 \cancel{\text{min}}}\right)(20 \cancel{\text{min}})$

$= 4800 \text{ ft}$

Woman $y = (5 \text{ ft/s})(15 \text{ min})\left(\frac{60 \text{ s}}{\text{min}}\right) = 4500 \text{ ft}$

$z = ?$ $\quad (x+y)^2 + (500)^2 = z^2$

$z = \sqrt{(x+y)^2 + (500)^2} = \sqrt{(4800+4500)^2 + (500)^2}$

$(x+y)(x'+y') = zz'$ Find z'

$$z' = \frac{(x+y)(x'+y')}{z}$$

$$z' = \frac{(4800+4500)(4+5)}{\sqrt{(4800+4500)^2+(500)^2}} \approx 8.987 = 8.99\,\frac{ft}{s}$$

$$\approx 9\,ft/s$$

§ 3.10 #37 $f(t) = \operatorname{sech}^2(e^t)$

$$f(t) = \left(\operatorname{sech}(e^t)\right)^2$$

$$f'(t) = 2\left(\operatorname{sech}(e^t)\right)\operatorname{sech}(e^t)\tanh(e^t)\cdot e^t$$

$$f'(t) = 2e^t \operatorname{sech}^2(e^t)\tanh(e^t)$$