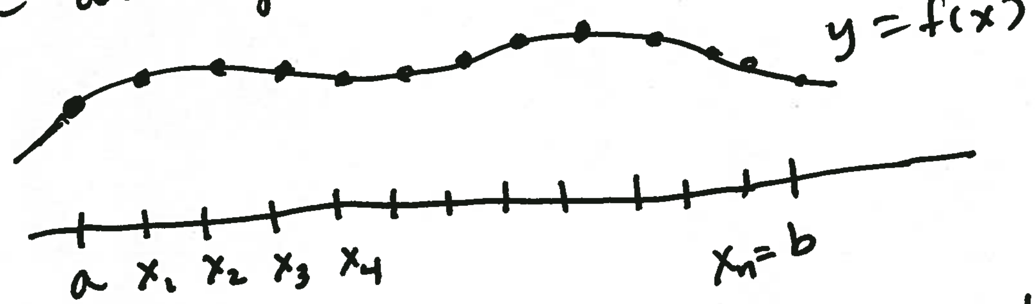
§ 6.5 The Average Value of a Function.

Hw §6.5 # 1–8

Suppose that f is a continuous function on [a, b]. We will define the average value of f on [a, b].



If we take n points, then the average of those values is

$$\frac{f(x_1)+f(x_2)+f(x_3)+\dots+f(x_n)}{n}$$

$$= \sum_{i=1}^{n} \frac{f(x_i)}{n} = \sum_{i=1}^{n} f(x_i) \underbrace{\frac{(b-a)}{n}}_{\Delta x} \frac{1}{(b-a)}$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\Delta x$$ We now let $n \to \infty$

~~We define~~

We define the average value of f on $[a,b]$ as

$$f_{AVE} = \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\boxed{f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Example: Find the average value of $f(x)=x^2$ on $[0,2]$.

SOLUTION

$$f_{AVE} = \frac{1}{2-0}\int_0^2 x^2\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{6}\left[2^3 - 0^3\right]$$

$$= \frac{8}{6} = \frac{4}{3}$$

The Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is, $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$

<u>Proof</u> Let $g(x) = \displaystyle\int_a^x f(t)\,dt.$

g is continuous and diff. on $[a,b]$ and by Fund Thm Calc Part 1,

$$g'(x) = f(x).$$

By the Mean Value Theorem, there exists c in (a,b) such that

$$g'(c) = \frac{g(b)-g(a)}{b-a}$$

$g'(c) = f(c)$

so $$f(c) = \frac{g(b)-g(a)}{b-a} = \frac{\int_a^b f(t)\,dt - \overbrace{\int_a^a f(t)\,dt}^{=0}}{b-a}$$

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt \quad \square$$