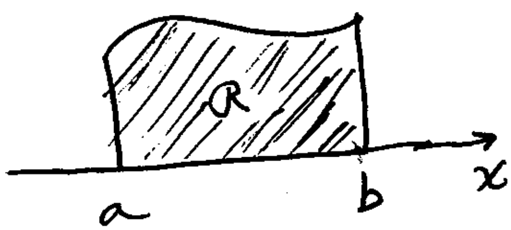
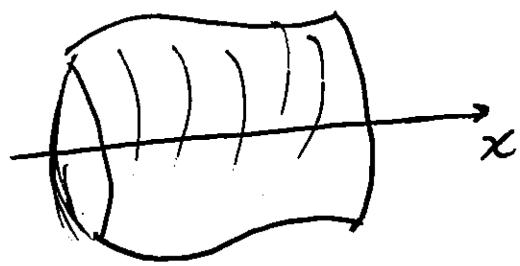
§6.2 Volumes of Solids

HW §6.2 #1-36

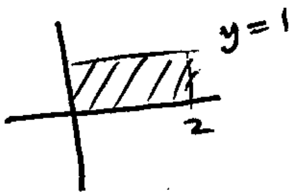
Solids of Rotation

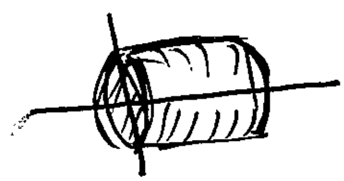


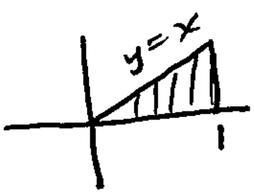


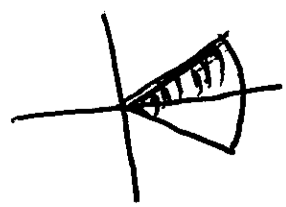
We rotate the region R about the x-axis

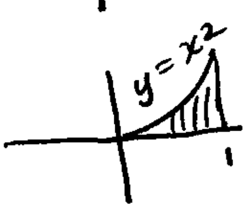
EXAMPLE: Sketch the solid obtained by rotating R about the x-axis.

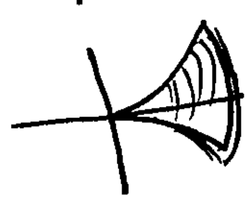
a) 

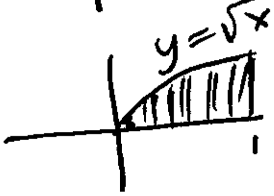


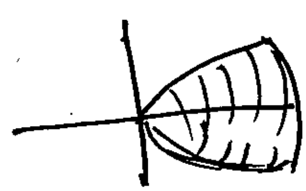
b) 



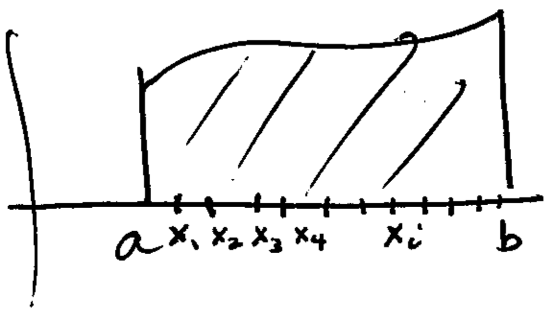
c) 

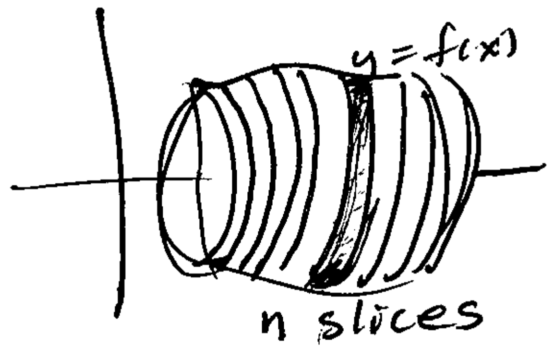


d) 

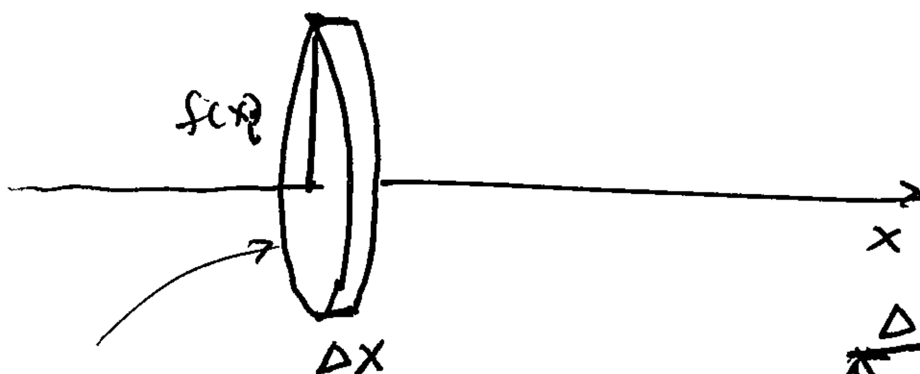


We now derive a formula for the volume of a solid found by rotating a region R about the x-axis.





The ith slice



Area of the face is πr^2, where the radius $r=f(x)$

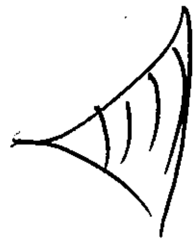
Area of ~~the rectangle~~ $= \pi\big(f(x_i)\big)^2$

Recall: ~~Area~~ Volume of Cylinder

$V=\pi r^2 h$

Volume of ith slice

$\approx \pi\big(f(x_i)\big)^2 \Delta x$



Approximate by cylindrical disks.

Now, let n, the number of slices, approach infinity.

Define volume as

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi (f(x_i))^2 \Delta x$$

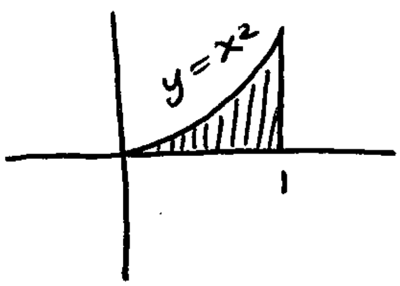
$$\boxed{V = \int_a^b \pi (f(x))^2 \, dx}$$

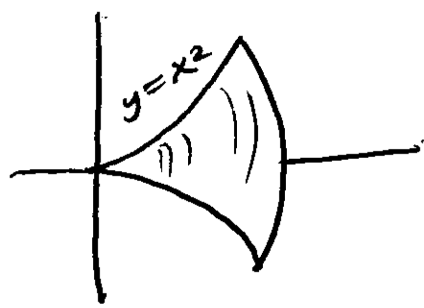
Back to example, $f(x) = x^2$, $a = 0$, $b = 1$.

$$V = \int_0^1 \pi (x^2)^2 \, dx = \int_0^1 \pi x^4 \, dx$$

$$= \pi \frac{x^5}{5} \Big|_0^1 = \pi \left(\frac{(1)^5}{5} - \frac{0^5}{5} \right) = \frac{\pi}{5}.$$

EXAMPLE Let R be the region bounded by $y = x^2$, the x-axis, $x=0$, and $x=1$. The region R is rotated about the x-axis. Find the volume.





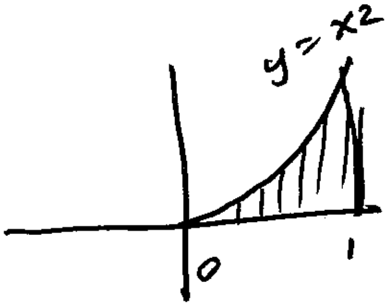
$$V = \int_a^b \pi (f(x))^2 \, dx$$

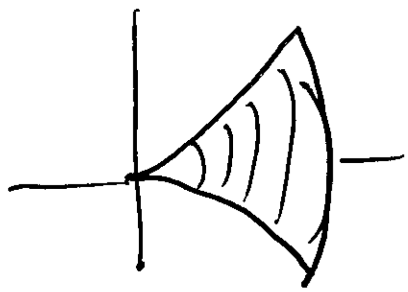
$$V = \int_0^1 \pi (x^2)^2 \, dx = \pi \int_0^1 x^4 \, dx$$

$$= \pi \frac{x^5}{5}\Big|_0^1 = \pi \left(\frac{1^5}{5} - \frac{0^5}{5} \right)$$

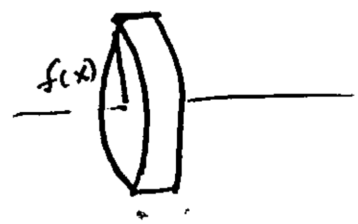
$$= \frac{\pi}{5}$$

EXAMPLE Find the volume of the solid found by rotating R about the x-axis where R is the the region bounded by $y=x^2$, the x-axis, $x=0$, and $x=1$.





one disk



dx

$$V = \int_a^b \pi (f(x))^2 \, dx$$

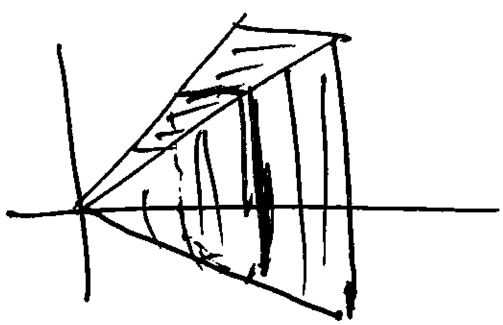
$$V = \int_0^1 \pi (x^2)^2 \, dx$$

$$V = \int_0^1 \pi x^4 \, dx$$

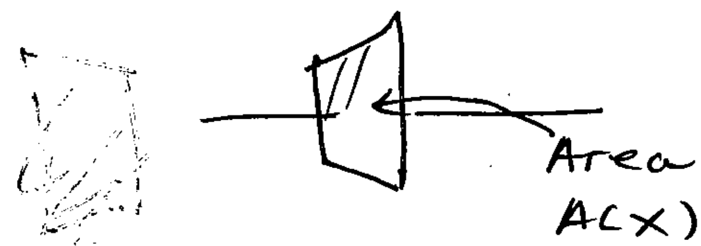
$$= \pi \frac{x^5}{5}\Big|_0^1 = \pi\left(\frac{1^5}{5} - \frac{0^5}{5}\right) = \frac{\pi}{5}$$

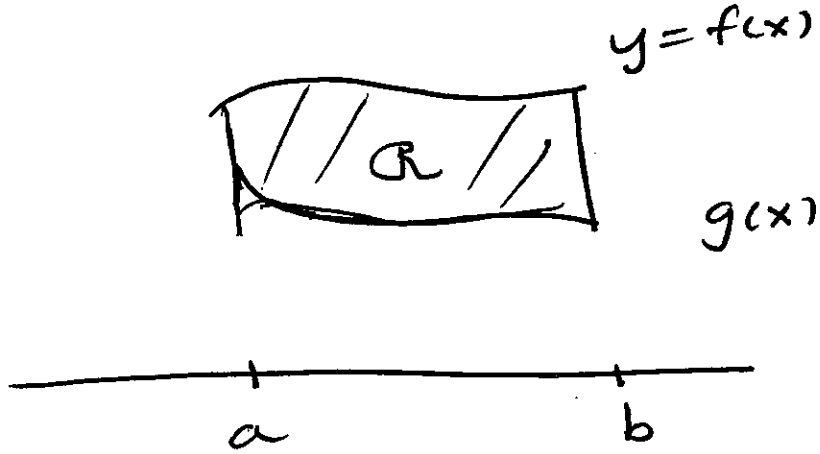
In general, the formula for volume is

$$V = \int_a^b A(x)\,dx$$ where $A(x)$ is the area of a cut through x.



Pyramid (not a solid of rotation)

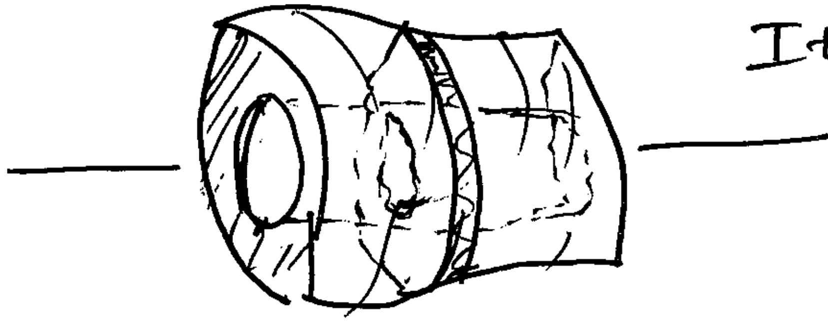




Suppose ℛ is bounded by

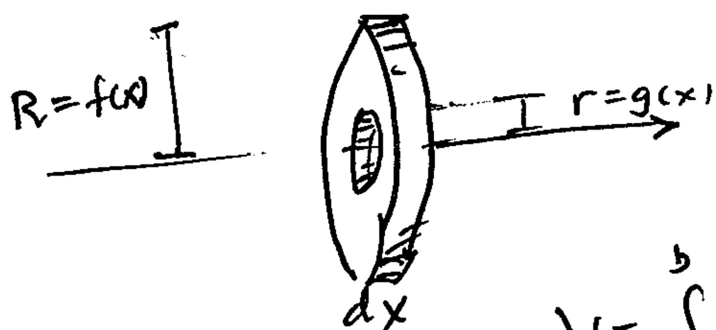
$y=f(x)$, $y=g(x)$, $x=a$, and, $x=b$.

Rotate about x-axis



It's a hollow solid.

Washere



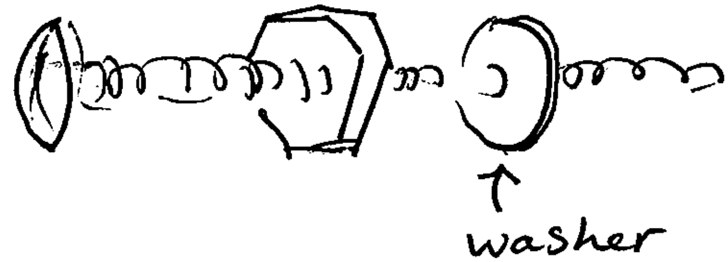
Area:

$\pi R^2 - \pi r^2$

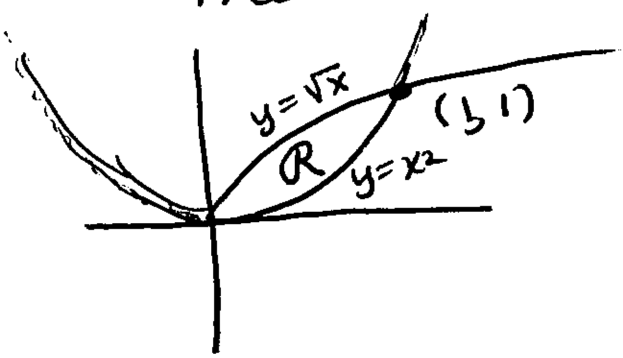
$= \pi\left(f(x)\right)^2 - \pi\left(g(x)\right)^2$

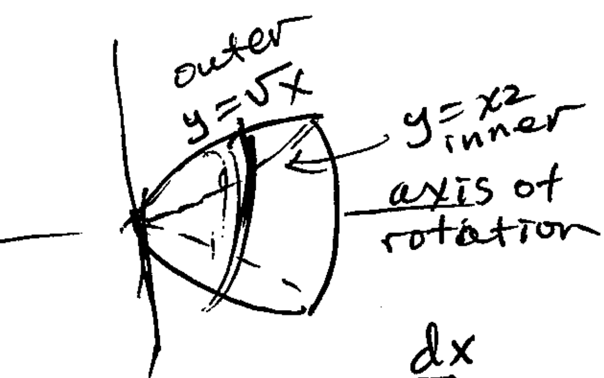
$$V = \int_a^b \left(\pi\Big(\underset{\text{outer}}{f(x)}\Big)^2 - \pi\Big(\underset{\text{inner}}{g(x)}\Big)^2\right) d$$

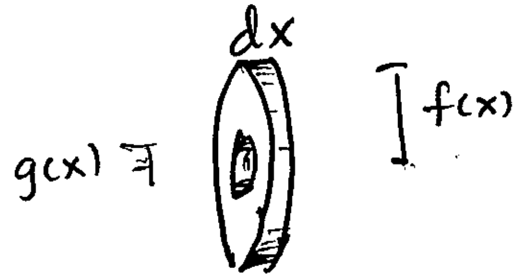
~~Ex~~



EXAMPLE The region R is bounded by $y=x^2$, and $y=\sqrt{x}$. Find the volume of the solid obtained by ~~rotation~~ rotating R about the x-axis.







$$V=\int_a^b\left(\pi\left(f(x)\right)^2-\pi\left(g(x)\right)^2\right)dx$$

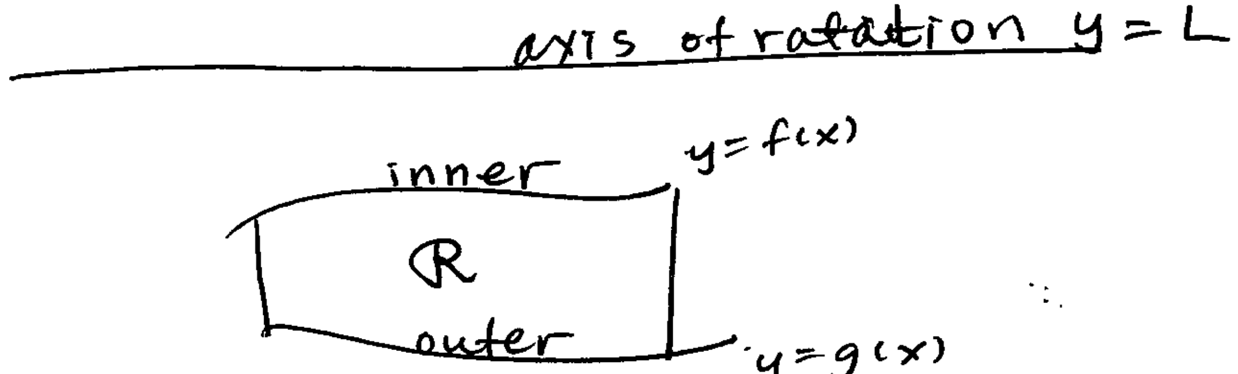
$$=\int_0^1\left(\pi\left(\sqrt{x}\right)^2-\pi\left(x^2\right)^2\right)dx$$

$$=\pi\int_0^1\left(x-x^4\right)dx=\pi\left(\frac{x^2}{2}-\frac{x^5}{5}\right)\Big|_0^1$$

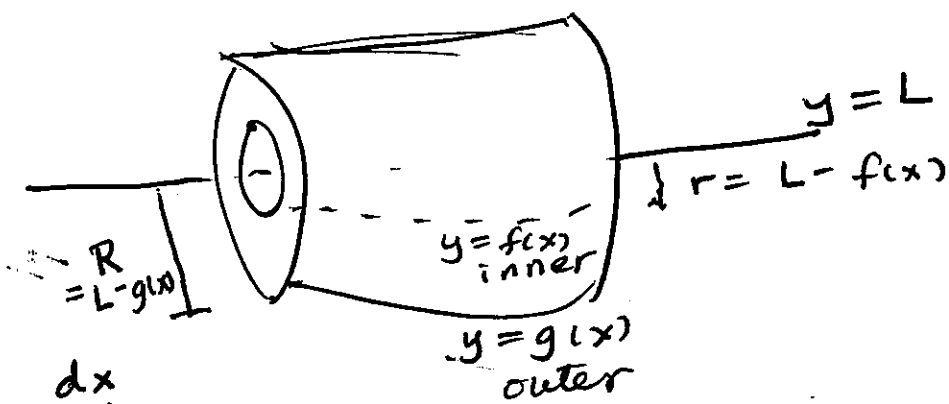
$$=\pi\left(\frac{1^2}{2}-\frac{1^5}{5}\right)-0$$

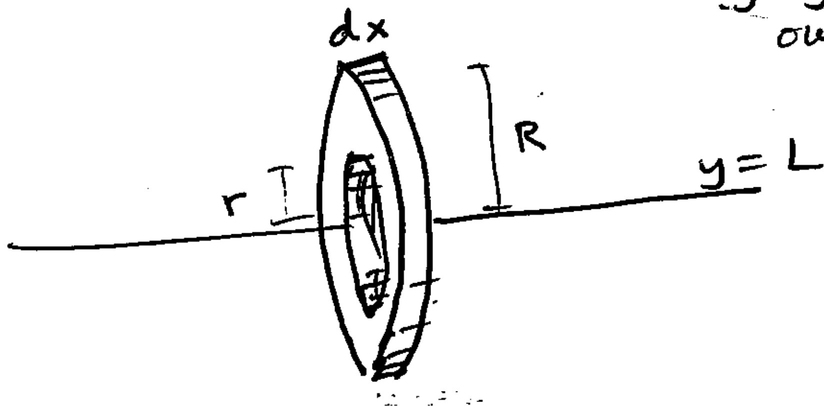
$$=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\pi\left(\frac{5}{10}-\frac{2}{10}\right)=\frac{3\pi}{10}$$

Rotating about the line $y = L$.



Rotate R about $y = L$



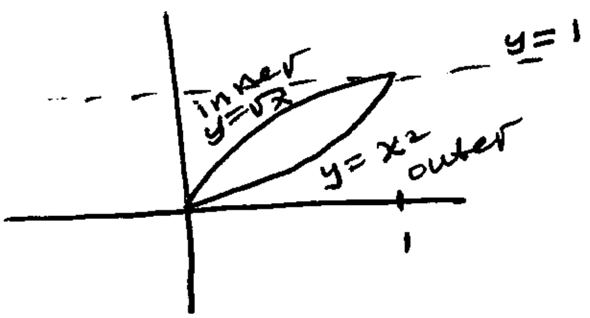


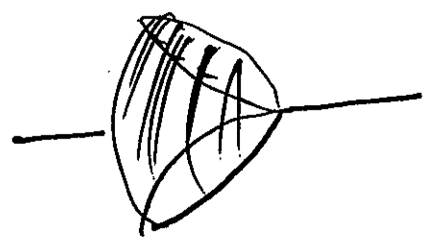
$$V = \int_a^b \left(\pi \big(\underbrace{L - \underbrace{g(x)}_{outer}}_{R} \big)^2 - \pi \big(\underbrace{L - \underbrace{f(x)}_{inner}}_{r} \big)^2 \right) dx$$

FORMULA FOR ROTATION ABOUT THE LINE $y = L$

$$V = \int_a^b \left[\pi\big(\underbrace{f(x)}_{\text{outer}} - L\big)^2 - \pi\big(\underbrace{g(x)}_{\text{inner}} - L\big)^2\right] dx$$

EXAMPLE $\mathcal{R}$ is the region bounded by $y = x^2$ and $y = \sqrt{x}$. The region $\mathcal{R}$ is rotated about the line $y = 1$. Find the volume of the resulting solid.





$$V = \int_0^1 \left[\pi\big(\underbrace{x^2}_{\text{outer}} - 1\big)^2 - \pi\big(\underbrace{\sqrt{x}}_{\text{inner}} - 1\big)^2\right] dx$$

$$= \pi \int_0^1 \left[\left((x^2)^2 - 2(x^2) + 1\right) - \left((\sqrt{x})^2 - 2\sqrt{x} + 1\right)\right] dx$$

$$= \pi \int_0^1 \left(x^4 - 2x^2 + 1 - x + 2x^{1/2} - 1\right) dx$$

$$= \pi \int_0^1 \left(x^4 - 2x^2 - x + 2x^{1/2}\right) dx$$

$$= \pi \left[\frac{x^5}{5} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2\frac{x^{3/2}}{3/2} \right]_0^1$$

$$= \pi \left[\frac{1^5}{5} - \frac{2}{3}\cdot 1^3 - \frac{1^2}{2} + 2\cdot\frac{2}{3}\cdot 1^{3/2} \right]$$

$$= \pi \left[\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3} \right]$$

$$= \pi \left[\frac{1}{5} + \frac{2}{3} - \frac{1}{2} \right] = \pi \left[\frac{6}{30} + \frac{20}{30} - \frac{15}{30} \right]$$

$$= \pi \cdot \frac{11}{30} = \cancel{\frac{\pi\, 11}{30}}$$

$$= \frac{11\pi}{30}$$