§5.5 Continued

EXAMPLE Evaluate the definite integral.

$$\int_0^4 x\sqrt{x^2+9}\,dx$$

Method I • First evaluate the indefinite integral.

$$\int x\sqrt{x^2+9}\,dx \qquad u = x^2+9$$

$$du = 2x\,dx$$

$$\int \sqrt{\underbrace{x^2+9}_{u}}\ \underbrace{x\,dx}_{\frac{1}{2}du} \qquad \tfrac{1}{2}du = x\,dx$$

$$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} \cancel{+C}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\,(x^2+9)^{3/2}$$

• Now evaluate at endpoints.

$$= \left[\frac{1}{3}(x^2+9)^{3/2}\right]_0^4 = \frac{1}{3}\left[(x^2+9)^{3/2}\right]_0^4$$

$$= \frac{1}{3}\left[(4^2+9)^{3/2} - (0^2+9)^{3/2}\right]$$

$$= \frac{1}{3}\left[(25)^{3/2} - (9)^{3/2}\right]$$

$$= \frac{1}{3}\left[\left(\sqrt{25}\right)^3 - \left(\sqrt{9}\right)^3\right] = \frac{1}{3}\left[125-27\right]$$

$$= \frac{98}{3}$$

Method II $\displaystyle\int_{x=0}^{x=4} x\sqrt{x^2+9}\,dx$

$u = x^2+9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

- when $x=0$

  $u = 0^2 + 9 = 9$

- when $x=4$

  $u = 4^2+9 = 16+9$

  $= 25$

$$\frac{1}{2}\int_{u=9}^{u=25} \sqrt{u}\,du$$

$$\frac{1}{2}\int_{9}^{25} u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2}\Bigg|_{9}^{25}$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_9^{25}$$

$$= \frac{1}{3}\left[(25)^{3/2} - (9)^{3/2}\right]$$

$$= \frac{1}{3}\left[125 - 27\right] = \frac{98}{3}$$

Example: Evaluate the definite integral.

(2) $$\int_0^1 x e^{-x^2} dx$$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2} du = x\,dx$

- when $x=0$, $u = -(0)^2 = 0$
- when $x=1$, $u = -(1)^2 = -1$

$$-\frac{1}{2} \int_0^{-1} e^u du$$

$$= -\frac{1}{2} e^u \Big|_0^{-1}$$

$$= -\frac{1}{2} [e^{-1} - e^0]$$

$$= -\frac{1}{2} \left[\frac{1}{e} - 1\right] = \frac{1}{2} \left[1 - \frac{1}{e}\right]$$

(2) $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$\displaystyle\int_0^{\pi/4} \underbrace{\sec x}_{u} \, \underbrace{(\sec x \tan x) \, dx}_{du}$

when $x = 0$

$u = \sec(0) = \dfrac{1}{\cos(0)}$

$= \dfrac{1}{1} = 1$

when $x = \pi/4$

$u = \sec \pi/4$

$= \dfrac{1}{\cos \frac{\pi}{4}} = \dfrac{1}{\frac{\sqrt{2}}{2}}$

$= \dfrac{2}{\sqrt{2}} =$

$= \dfrac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \sqrt{2}$

$\displaystyle\int_1^{\sqrt{2}} u^1 \, du = \frac{u^2}{2} \Big|_1^{\sqrt{2}}$

$= \frac{1}{2}\left[(\sqrt{2})^2 - (1)^2\right]$

$= \frac{1}{2}(2-1) = \frac{1}{2}$

OR

$\displaystyle\int_0^{\pi/4} \tan x \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

x	u
0	$\tan 0 = 0$
$\frac{\pi}{4}$	$\tan \frac{\pi}{4} = 1$

$\displaystyle\int_0^1 u^1 \, du = \frac{u^2}{2} \Big|_0^1$

$= \dfrac{1^2}{2} - \dfrac{0^2}{2} = \dfrac{1}{2}$

Formula $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

③ $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

$u = \frac{1}{x} = x^{-1}$

$du = -1 \cdot x^{-2}\,dx$

$du = -\frac{1}{x^2}\,dx$

$-du = \frac{1}{x^2}\,dx$

x	$u = 1/x$
1	1
2	$1/2$

$= \displaystyle\int_1^2 e^{\overset{u}{1/x}} \cdot \underbrace{\frac{1}{x^2}\,dx}_{-du}$

$= -\displaystyle\int_1^{1/2} e^u\,du$

$= \displaystyle\int_{1/2}^{1} e^u\,du$

$= \left[e^u\right]_{1/2}^{1} = e^1 - e^{1/2}$

$= e - \sqrt{e}$

EXAMPLE $\int \frac{x+1}{x^2+1}\,dx$

$$= \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

I $\int \frac{x}{x^2+1}\,dx$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \int \underbrace{\left(\frac{1}{x^2+1}\right)}_{u} \cdot \underbrace{x\,dx}_{\frac{1}{2}du}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C_1$$

$$= \frac{1}{2}\ln|x^2+1| + C_1$$

II $\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C_2$

$\boxed{\text{Answer: } \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + C}$

Example $\int x\sqrt{x+1}\,dx$

$u = x+1 \Rightarrow u-1 = x$

$du = dx$

similar to §5.5 #44-46

$$\int \underbrace{x}_{u-1}\sqrt{\underbrace{x+1}_{u}}\,\underbrace{dx}_{du}$$

$$= \int (u-1)\sqrt{u}\,du$$

$$= \int (u-1)u^{1/2}\,du$$

$$= \int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

§5.5 # 36 $\int \frac{\sin x}{1+\cos^2 x}\,dx$

$$= \int \frac{1}{1+(\cos x)^2} \cdot \underbrace{\sin x\,dx}_{-du}$$

$$= -\int \frac{1}{1+u^2}\,du$$

$$= -\tan^{-1}(u) + C$$

$$= -\tan^{-1}(\cos x) + C$$

~~$u = 1+\cos^2 x$~~
~~$u = 1+(\cos x)^2$~~
~~$du = 2(\cos x)(-\sin x)d$~~
~~dead end.~~

~~$u = \sin x$~~
~~$du = \cos x\,dx$~~
~~dead end~~

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

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$$\int \frac{\tan^{-1}x}{1+x^2}\,dx$$

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}\,dx$

$$= \int (\tan^{-1}x)\left(\frac{1}{1+x^2}\right)dx$$

$$= \int u\,du = \frac{u^2}{2} + C$$

$$= \frac{(\tan^{-1}x)^2}{2} + C$$