§5.2 Continued

Comparison Properties of the Integral.

① If $f(x) \geq 0$ on $a \leq x \leq b$,

then $\int_a^b f(x)\,dx \geq 0$.

② If $f(x) \leq g(x)$ on $a \leq x \leq b$,

then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

__Pf__ $g(x) \geq f(x)$ on $[a, b]$

implies $g(x) - f(x) \geq 0$

$$\Rightarrow \int_a^b (g(x) - f(x))\,dx \geq 0$$

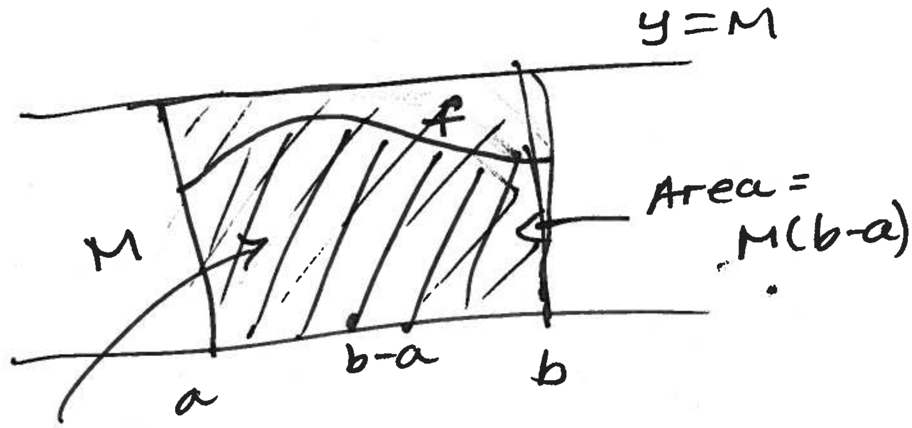
$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0$$

$$\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$$

③ If $m \le f(x) \le M$ for $a \le x \le b$

then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$



area below f is

$$\int_a^b f(x)\,dx$$

§5.3 The Fundamental Theorem of Calculus

HW §5.3 # 2-42

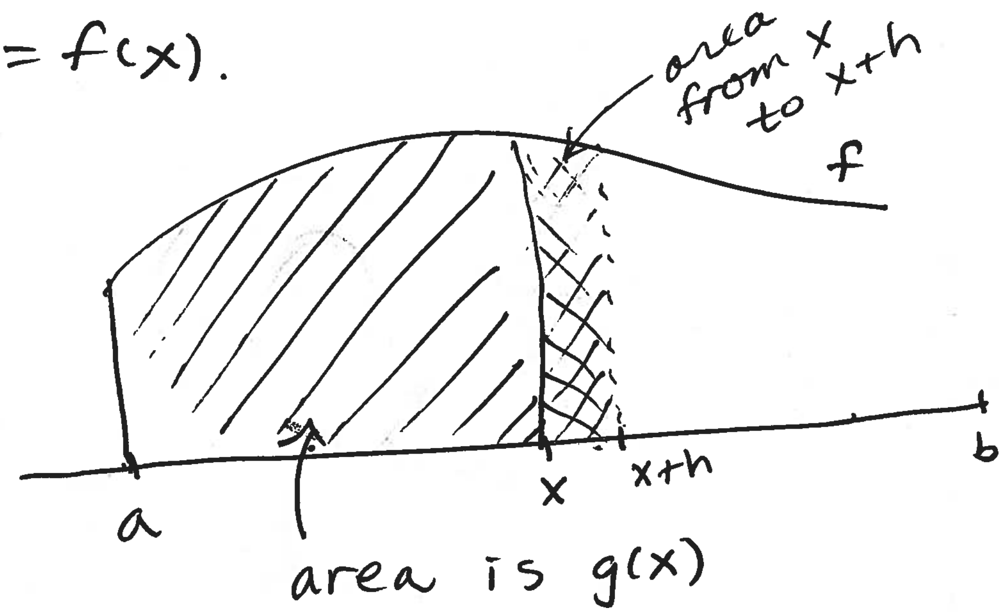
The Fundamental Theorem of Calculus

Part 1: If f is continuous on $[a, b]$, then the function defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$

is continuous on $[a, b]$ and

$g'(x) = f(x)$.

Proof

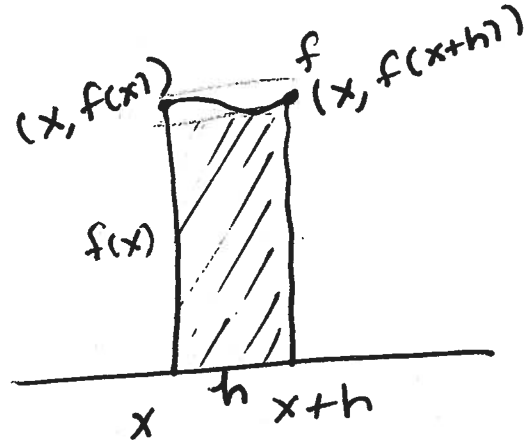


$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(g(x+h) - g(x) \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt \right)$$

$$g'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$



<u>Idea</u>

This is almost a rectangle.

Area = base × height

$= h * f(x)$

Area is also given by $\int_x^{x+h} f(t)\,dt$.

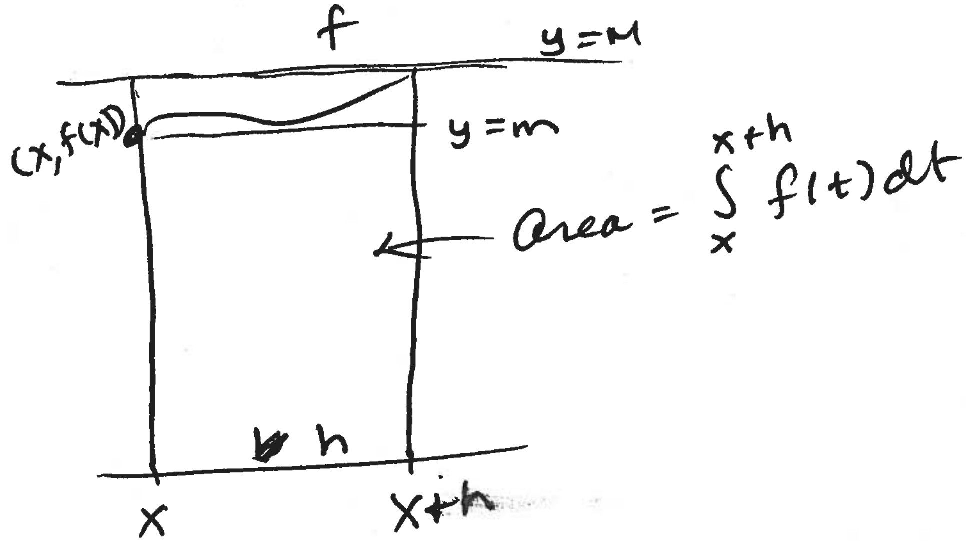
So for h small

$$h \cdot f(x) \approx \int_x^{x+h} f(t)\,dt$$

$$f(x) \approx \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

Let M = max of f on $[x, x+h]$

m = min of f on $[x, x+h]$



$$m \le f(x) \le M$$

$$mh \le \int_x^{x+h} f(x) \le M \cdot h$$

$$m \le \frac{1}{h}\int_x^{x+h} f(x) \le M$$

Note that m and M are functions of h.

$$\lim_{h\to 0} m = f(x)$$

$$\lim_{h\to 0} M = f(x)$$

by Squeeze Theorem

$$g'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

$\square$

The Fundamental Theorem of Calculus, Part 2.

If f is continuous on $[a, b]$,

then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

<u>Proof</u> $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of f, by the Fundamental Theorem of Calculus, Part 1. If F is also an antiderivative, then $F(x) = g(x) + C$ for some constant C.

We have

$$\begin{aligned} F(b) - F(a) &= (g(b) + C) - (g(a) + C) \\ &= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0} \\ &= \int_a^b f(t)\,dt \end{aligned}$$

So

$$F(b) - F(a) = \int_a^b f(x)\,dx \quad \checkmark$$

EXAMPLE Evaluate (using the Fundamental Theorem of Calculus Part 2.)

a) $\int_0^1 x^2\,dx$

$f(x) = x^2$

$F(x) = \frac{x^3}{3}$

$a = 0,\ b = 1$

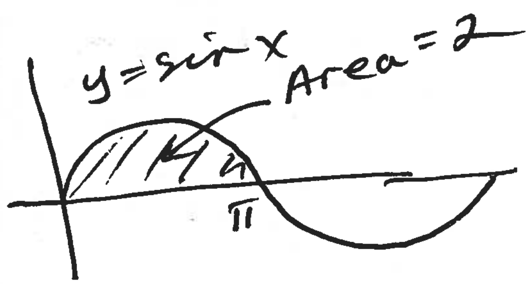
$F(b) - F(a) = F(1) - F(0)$

$= \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$

Here's a shorter way to write this.

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$$

b) $$\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = (-\cos\pi) - (-\cos 0)$$
$$= -(-1) + 1 = 2$$



c) $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = \Big[\tan x\Big]_0^{\pi/4}$

$= \tan\frac{\pi}{4} - \tan 0$

$= 1 - 0 = 0$

d) $\displaystyle\int_1^e \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^e = \ln e - \ln 1$

$= 1 - 0 = 1$

You can now do §5.3 #19-41.

e) $\displaystyle\int_{-1}^{2} \frac{1}{x}\,dx$ $\qquad$ $y = \frac{1}{x}$ is not continuous at $x = 0$.

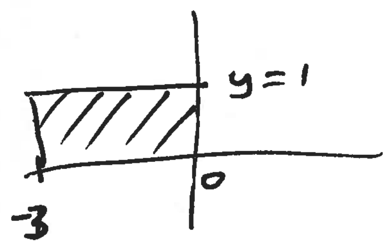
<u>DNE</u>

§5.2 #37 $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx = \overset{\mathrm{I}}{\int_{-3}^{0} 1\,dx} + \overset{\mathrm{II}}{\int_{-3}^{0} \sqrt{9-x^2}\,dx}$

Evaluate by interpreting in terms of area.

$y = 1 + \sqrt{9-x^2}$

I $\int_{-1}^{0} 1\,dx = 1(0-{-3}) = 1\cdot 3 = 3$

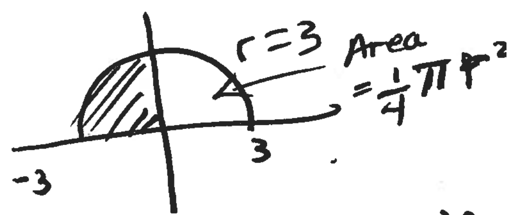


II First

$y = \sqrt{9-x^2}$, $y \geq 0$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$



$\int_{-3}^{0} \sqrt{9-x^2}\,dx = \frac{\pi(3)^2}{4} = \frac{9\pi}{4}$

$= \boxed{3 + \frac{9}{4}\pi}$

§5.2 #35 Evaluate the integral by interpreting it in terms of areas.

$$\int_0^3 \left(\frac{1}{2}x-1\right)dx = \frac{1}{4} - 1 = \frac{-3}{4}$$

SOLUTION $y = \frac{1}{2}x - 1$