§5.2 The Definite Integral

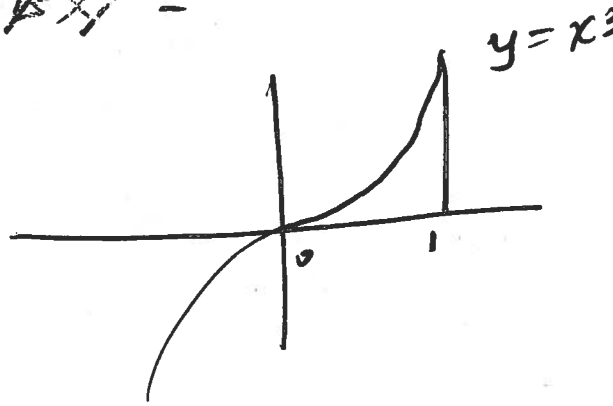
Homework

§5.2 # 17-20, 29, 30, 33-40

Area under the curve is defined

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \quad a \le x \le b$$

<u>Example</u>: Evaluate the area below the curve $y = x^3$, $0 \le x \le 1$.



$$\text{Area} = \int_0^1 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$0 \le x \le 1$$

$$a=0,\ b=1$$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x$$
$$= 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3\cdot n}$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

Use formula $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{4n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n^2+2n+1}{4n^2}\right)^{1/n^2}_{1/n^2} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$$

$$= \frac{1}{4}$$

§5.2 #22 Evaluate the integral.

$$\int_1^4 (x^2+2x-5)\,dx$$

SOLUTION

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \qquad 1 \le x \le 4$$

$$f(x) = x^2+2x-5$$

$$a = 1$$

$$b = 4$$

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = 1 + i\cdot\frac{3}{n} = 1 + \frac{3i}{n}$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\cdot\frac{3}{n}}$$ ← Stop here for test.

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + 2\cdot\frac{3i}{n} + \left(\frac{3i}{n}\right)^2\right) + \left(2+\frac{6i}{n}\right) - 5\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{9i^2}{n^2} + \frac{12i}{n} - 2\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^2 + \frac{36}{n^2} i - \frac{6}{n}\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{6}{n}\sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty} \frac{\overset{9}{\cancel{27}}}{n^{\cancel{3}2}} \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}2} + \frac{36}{n^2} \cdot \frac{n(n+1)}{2} - \frac{6}{n}(1 \cdot n)$$

$$= \lim_{n\to\infty} \frac{9}{2} \frac{(n+1)(2n+1)}{n^2} + 18 \frac{n+1}{n} - 6$$

$$= \lim_{n\to\infty} \frac{9}{2}\left(\frac{2n^2+3n+1}{n^2}\right) + 18\left(\frac{n+1}{n}\right) - 6$$

$$= \frac{9}{2} \frac{2}{1} + 18 \cdot \frac{1}{1} - 6 = 9 + 18 - 6$$

$$= 27 - 6 = \boxed{21}$$

Example (Similar to §5.2 #17-20)

Express the limit as a definite integral on the given interval.

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{\sin x_i}{x_i} \Delta x, \quad [\pi, 2\pi]$$

Solution

$$\int_{\pi}^{2\pi} \frac{\sin x}{x}\, dx$$

EXAMPLE (Similar to §5.2 # 29, 30)

Express the integral as a limit of Riemann sums. Do not evaluate the limit.

$$\int_0^{\pi} \cos 5x\, dx$$

SOLUTION

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$a = 0$

$b = \pi$

$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$

$x_i = a + i\Delta x = 0 + i \cdot \frac{\pi}{n}$
$= \frac{\pi i}{n}$

$f(x) = \cos 5x$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \cos 5\Big(\underbrace{\frac{i\pi}{n}}_{x_i}\Big) \cdot \underbrace{\frac{\pi}{n}}_{\Delta x}}$$

Propertres of the Definite Integral.

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

<u>Pf</u> $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$\Delta x = \frac{b-a}{n}$

$= -\frac{(a-b)}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(-\frac{(a-b)}{n}\right)$

$= -\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\frac{(a-b)}{n}$ $\quad \Delta x = \frac{a-b}{n}$

$= -\int_b^a f(x)\,dx$

② $\int_a^a f(x)\,dx = 0$

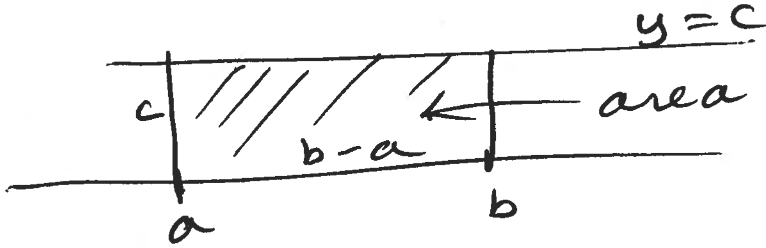
Proof

$$\int_a^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \Delta x = \frac{a-a}{n} = 0$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot 0$$

$$= 0$$

③ $\int_a^b c\,dx = c\,(b-a)$ c is a constant.



area = base × height
= (b−a)c

Example $\int_0^{\pi} 5\,dx = 5(\pi - 0) = 5\pi$

④ $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Proof : $\int_a^b (f(x)+g(x))\,dx$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i)+g(x_i))\Delta x$$

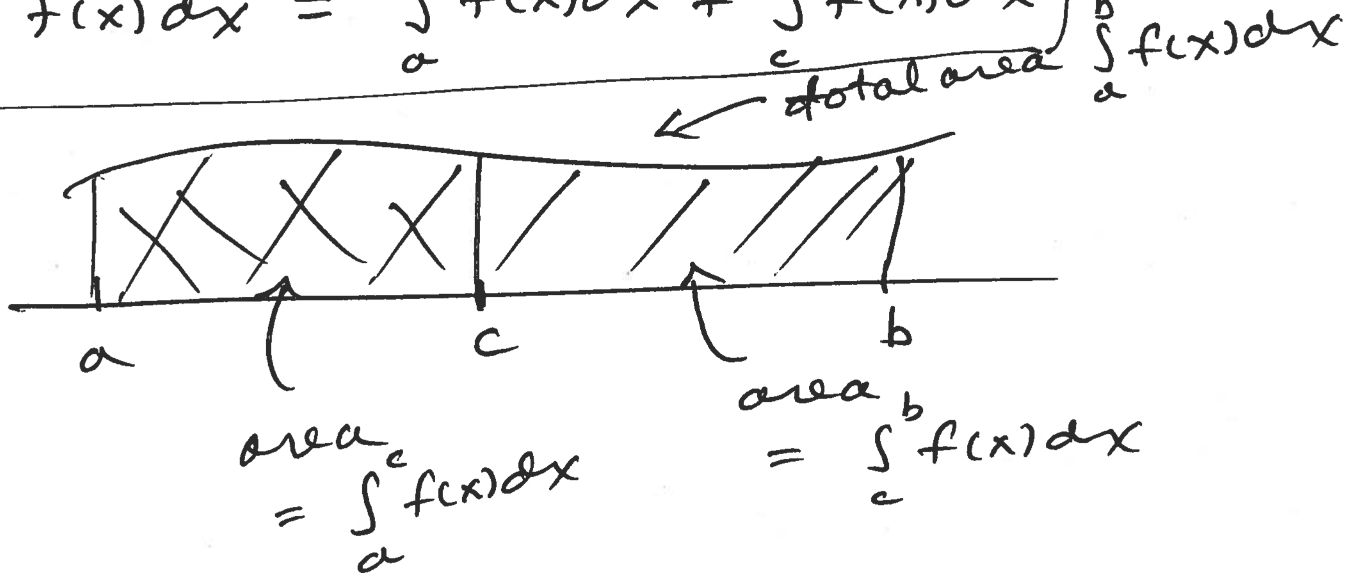
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x + \lim \sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

⑤ $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$ c is a constant.

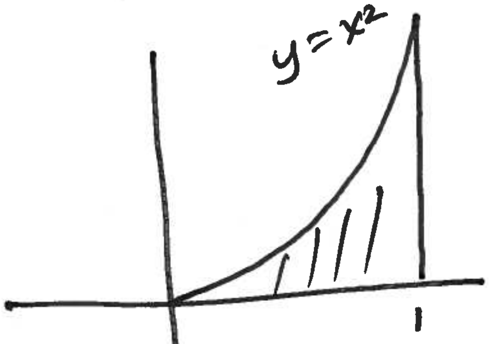
⑥ $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

⑦ $\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$



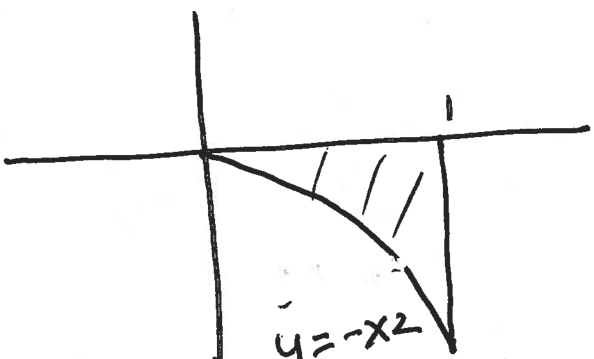
The Definite Integral and Area.

Explanation



$$\text{Area} = \int_0^1 x^2\,dx = \frac{1}{3}$$

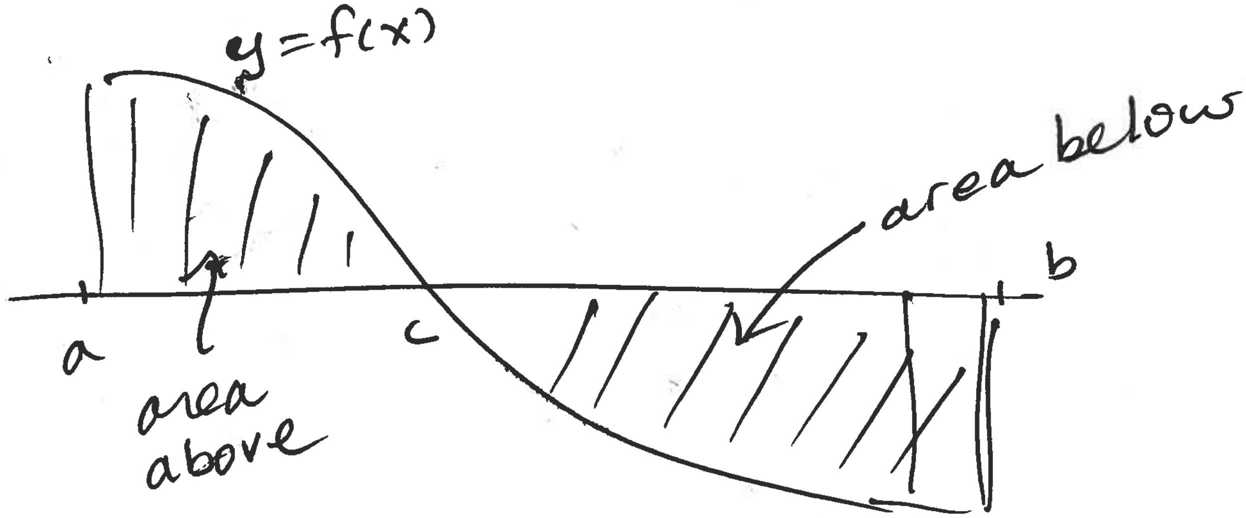
↑ we did this in a previous example



$$\text{Area} = \frac{1}{3}$$

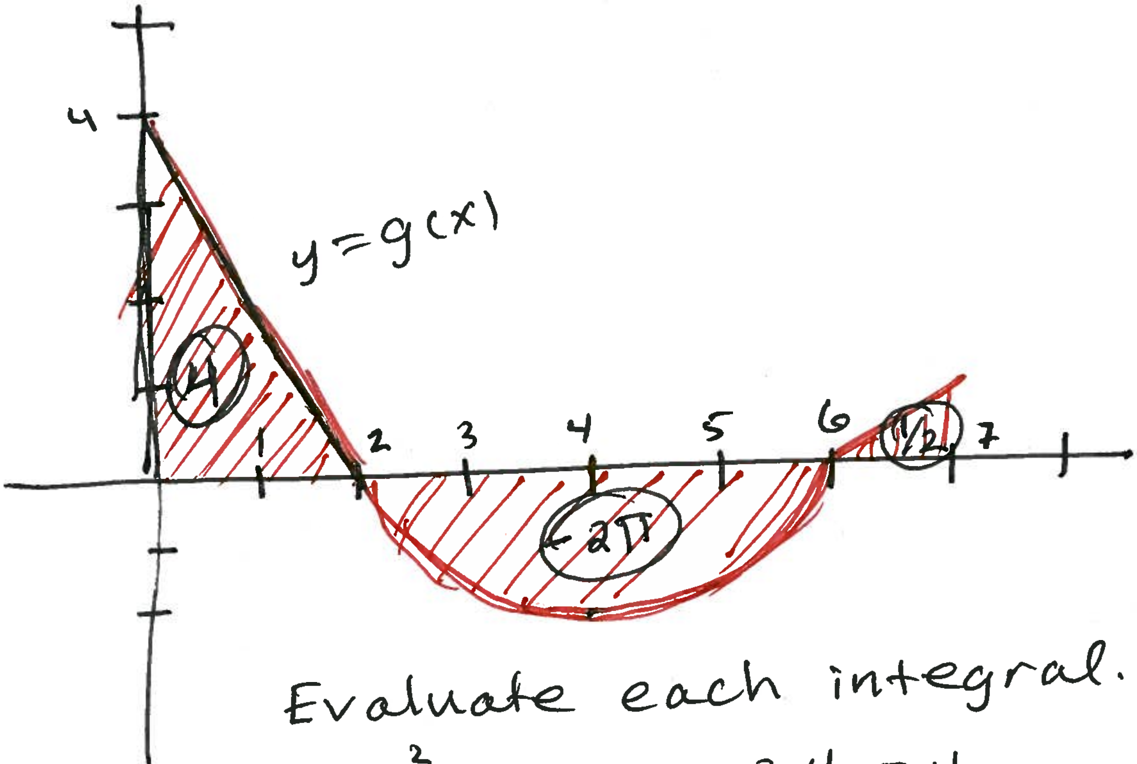
$$\int_0^1 (-x^2)\,dx = \int_0^1 -1\cdot x^2\,dx$$
$$= -1\cdot \int_0^1 x^2\,dx$$
$$= -1\cdot \frac{1}{3} = -\frac{1}{3} = -\text{Area}$$

If $f(x) \le 0$ on $a \le x \le b$,
then $\int_a^b f(x)\,dx = -$ Area between curve an x-axis.



$$\int_a^b f(x)\,dx = \text{Area above} - \text{Area below}.$$

Example



Evaluate each integral.

(a) $\int_0^2 g(x)\,dx = \frac{2\cdot 4}{2} = 4$

Area $\Delta = \frac{bh}{2}$

(b) $\int_2^6 g(x)\,dx = -(\pi(2)^2)\frac{1}{2} = -2\pi$

Area Circle $= \pi r^2$

$r = 2$

c) $\int_0^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = \boxed{\frac{9}{2} - 2\pi}$

$\int_6^7 g(x)\,dx = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$

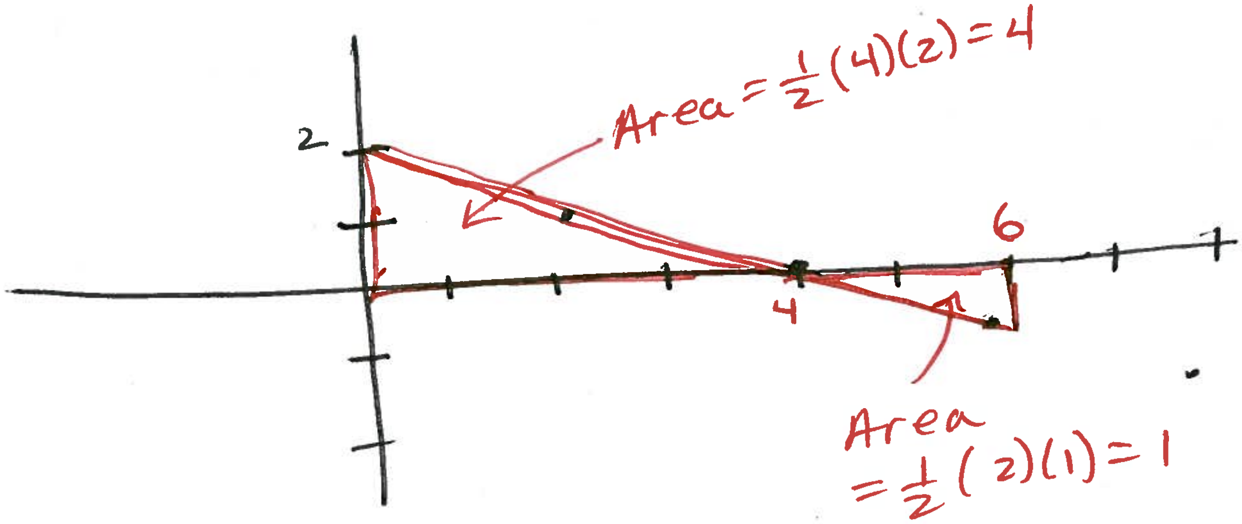
Example : Evaluate the integral by interpreting it in terms of area.

Area Above − Area below.

(1) $\int_0^6 \left(-\frac{1}{2}x+2\right)dx = 4-1=3$

SOLUTION $f(x) = -\frac{1}{2}x+2$

$m = -\frac{1}{2},\ b=2$



(2) $\int_{-2}^{2} \sqrt{4-x^2}\, dx = 2\pi$

<u>SOLUTION</u>

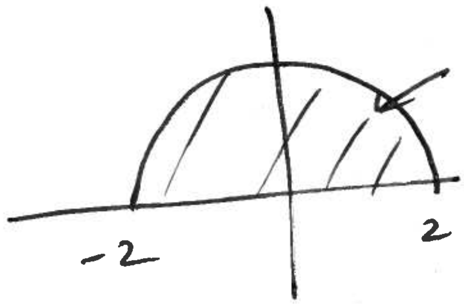
$y = \sqrt{4-x^2}$

$y^2 = 4 - x^2, \quad y \geq 0$

$x^2 + y^2 = 4$

upper semicircle

radius $r = 2$



Area $= \pi r^2/2$ ← half of circle

$= \dfrac{\pi(2)^2}{2} = 2\pi$