§5.1 Area beneath a Curve.

Homework §5.1 # 17 - 21

Σ sigma

Sums

$1^2 + 2^2 + 3^2 + \dots + 100^2$

$= \sum_{i=1}^{100} i^2$ ← this is summation notation.

Example Write in sigma notation.

(i) $2 + 4 + 6 + 8 + 10$

$= 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5$

$= \sum_{i=1}^{5} 2i$

(ii) $1 + 2 + 4 + 8 + 16 + 32$

$= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$= \sum_{i=0}^{5} 2^i$

example Find the sum.

(a) $\sum_{i=1}^{4}(2i+3) = (2\cdot 1+3) + (2\cdot 2+3) + (2\cdot 3+3) + (2\cdot 4+3)$

$= 5 + 7 + 9 + 11$

$= 32$

(b) $\sum_{i=0}^{3} 10^i = 10^0 + 10^1 + 10^2 + 10^3$

$= 1 + 10 + 100 + 1000$

$= 1{,}111$

Properties Suppose that c is a constant.

① $\cancel{\sum_{n} (a_n + b_n)}$ $\sum_{i=1}^{n}(a_i + b_i)$

$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Proof $\cancel{\sum_{n} (a_n + b_n)}$ $\sum_{i=1}^{n}(a_i + b_i)$

$= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \dots + (a_n + b_n)$

$= (a_1 + a_2 + a_3 + \dots + a_n) + (b_1 + b_2 + b_3 + \dots + b_n)$

$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Example: Find the sum.

(i) $\sum_{i=1}^{4}(2i+3) = (2\cdot1+3)+(2\cdot2+3)+(2\cdot3+3)$
$+(2\cdot4+3)$
$= 5+7+9+11$
$= 32$

(ii) $\sum_{i=0}^{3} 2^i = 2^0+2^1+2^2+2^3$
$= 1+2+4+8$
$= 15$

Properties Suppose that c is a constant.

1. $\sum_{i=1}^{n}(a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

example: $\sum_{i=1}^{n}(i^2+i) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$.

Pf $\sum_{i=1}^{n}(a_i+b_i) = (a_1+b_1)+(a_2+b_2)+\cdots(a_n+b_n)$
$= (a_1+a_2+a_3+\cdots a_n)+(b_1+b_2+\cdots b_n)$
$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

2. $\sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$

example: $\sum_{i=1}^{5} 3i^2 = 3\sum_{i=1}^{5} i^2$

Pf $\displaystyle\sum_{i=1}^{n} c\,a_i = ca_1 + ca_2 + ca_3 + \dots + ca_n$

$\displaystyle = c(a_1 + a_2 + \dots a_n)$

$\displaystyle = c\sum_{i=1}^{n} a_i \,.$

FORMULAS c is a constant

1. $\displaystyle\sum_{i=1}^{n} c = nc$

Pf $\displaystyle\sum_{i=1}^{n} c = \underbrace{c + c + c + \dots + c}_{n \text{ times}} = nc$

Example $\displaystyle\sum_{i=1}^{4} 5 = 4 \cdot 5 = 20$

2. $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \dots + n$

$\displaystyle = \frac{n(n+1)}{2}$

Pf (let's do the case where $n = 100$)

$1 + 2 + 3 + \dots + 49 + 50 + 51 + 52 + \dots + 98 + 99 + 100$

101

101

101

101

101

$\displaystyle = 101 \cdot 50 = \frac{(100+1)(100)}{2}$

$\displaystyle = \frac{(n+1)n}{2}$

when $n = 100$

3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

<u>example</u>: Find the sum.

a) $$\sum_{i=1}^{50} i^2 = 1^2 + 2^2 + 3^2 + \ldots + 50^2$$

$n = 50$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{50(50+1)(2\cdot 50+1)}{6}$$

$$= 42{,}925$$

b) $$\sum_{i=1}^{100} (3i^2 + 5i + 7)$$

$$= \sum_{i=1}^{100} 3i^2 + \sum_{i=1}^{100} 5i + \sum_{i=1}^{100} 7$$

$$= 3\sum_{i=1}^{100} i^2 + 5\sum_{i=1}^{100} i + \sum_{i=1}^{100} 7$$

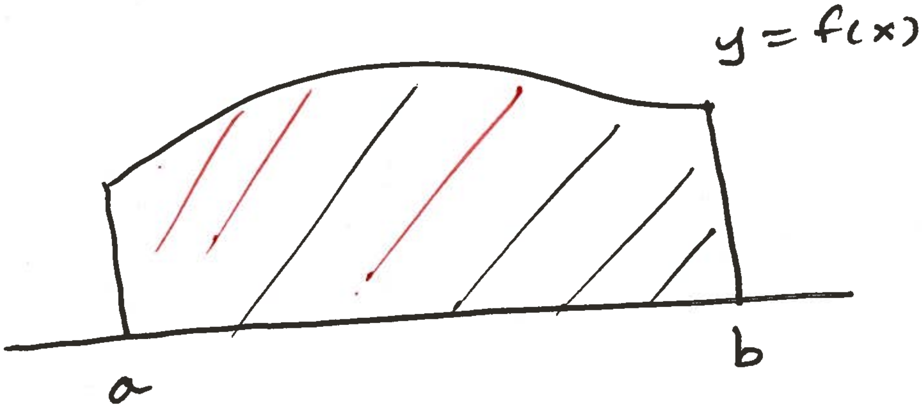
$n = 100$

$$= 3\,\frac{(100)(100+1)(2\cdot 100+1)}{6} + 5\,\frac{(100)(100+1)}{2} + 7\cdot 100$$

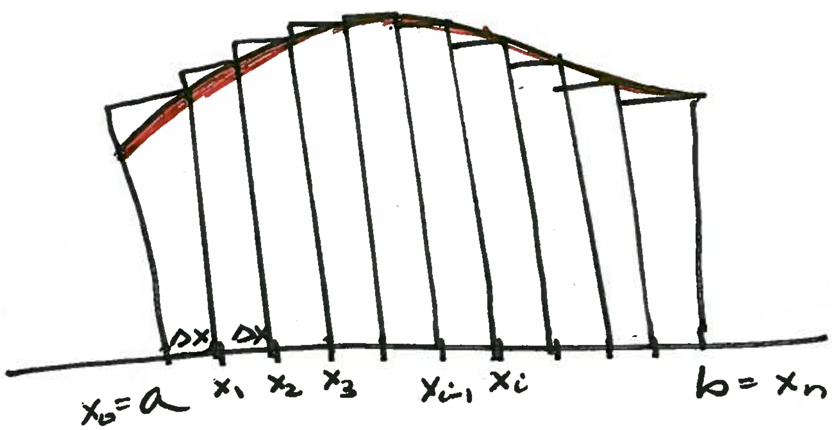
$$\frac{n(n+1)(2n+1)}{6} \qquad \frac{n(n+1)}{2} \qquad nc$$

$$= 1{,}041{,}000$$

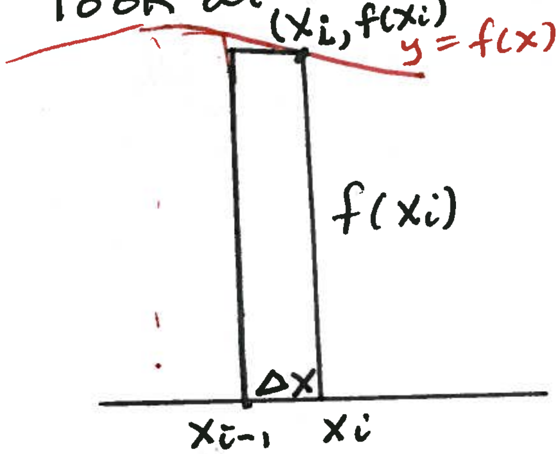
Areas under a Curve



Goal: To find the area of the region beneath a continuous function.



Let's look at the ith rectangle



Area = base × height

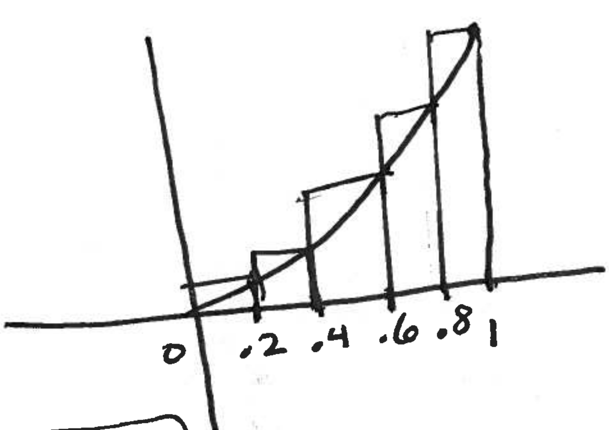
$$A_i = f(x_i)\Delta x$$

If we add together all n rectangles, we get

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

This is called a Riemann sum. It is an approximation for the area beneath the curve.

<u>Example</u> Find R_5 for $f(x) = x^2$, where $0 \le x \le 1$.



$n = 5$
$a = 0$
$b = 1$
$\Delta x = \frac{1}{5} = .2$

$x_0 = 0$
$x_1 = \frac{1}{5} = .2$
$x_2 = \frac{2}{5} = .4$
$x_3 = \frac{3}{5} = .6$
$x_4 = \frac{4}{5} = .8$
$x_5 = \frac{5}{5} = 1$

$$R_5 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$
$$= f(1/5)(\tfrac{1}{5}) + f(\tfrac{2}{5})(\tfrac{1}{5}) + f(\tfrac{3}{5})(\tfrac{1}{5}) + f(4/5)(1/5) + f(1)(1/5)$$
$$= (\tfrac{1}{5})^2(\tfrac{1}{5}) + (\tfrac{2}{5})^2(\tfrac{1}{5}) + \dots (1)^2(\tfrac{1}{5})$$
$$= \frac{1}{5}\left[\frac{1}{5^2} + \frac{2^2}{5^2} + \frac{3^2}{5^2} + \frac{4^2}{5^2} + \frac{5^2}{5^2}\right]$$
$$= \frac{1}{5}\cdot\frac{1}{5^2}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2\right]$$

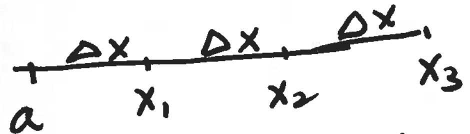
$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{5^3}\,\frac{(5)(5+1)(2\cdot 5+1)}{6}$$

$$= \frac{1}{5^3}\cdot\frac{5(6)(11)}{6} = \frac{11}{5^2} = \frac{11}{25}$$

FORMULAS: $\Delta x = \dfrac{b-a}{n}$

$x_i = a + i\Delta x$



$x_3 = a + 3\Delta x$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Example: Let $f(x) = x^2$, $0 \le x \le 1$.
Find a formula for R_n.

$a=0$, $b=1$

$\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{n} = \dfrac{1}{n}$

$x_i = a + i\Delta x = 0 + i\dfrac{1}{n} = \dfrac{i}{n}$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2\cdot\frac{1}{n}$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^2}\cdot\frac{1}{n}$$

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

no use summation formula.

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6n^2}.$$

$$R_5 = \frac{(5+1)(2\cdot5+1)}{6\cdot5^2} = \frac{(6)(11)}{6(25)} = \frac{11}{25} = .44$$

$$R_{10} = \frac{(10+1)(2\cdot10+1)}{6(10)^2} = \frac{(11)(21)}{6\cdot(100)} = 0.385$$

$R_{100} = 0.33835$

$R_{1000} = 0.3338335$

$R_{10000} = 0.33383335$

The idea is that the greater n is, the better the estimate.

we see that Area ≈ 0.33383335

Let's find the limit of R_n as $n \to \infty$.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2+n+2n+1}{6n^2}$$

$$= \lim_{n\to\infty} \left(\frac{2n^2+3n+1}{6n^2}\right) \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{2+\overset{\to 0}{3/n}+\overset{\to 0}{1/n^2}}{6} = \frac{2}{6} = \frac{1}{3}$$

We define area under the curve as the limit of the Riemann sum, R_n, as $n \to \infty$.

$$\text{Area} = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\Delta x}_{R_n}, \quad a \le x \le b.$$

We call this expression the definite integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

This is equal to area.