§4.9 Find f.

$$f''(t) = 2e^t + 3\sin t, \quad f(0)=0, \quad f(\pi)=0$$

$$f'(t) = 2e^t - 3\cos t + C$$

$$f(t) = 2e^t - 3\sin t + Ct + D$$

$$0 = f(0) = 2e^0 - 3\underbrace{\sin 0}_{=0} + C(0) + D$$

$$0 = 2 + D, \quad \boxed{D = -2}$$

$$0 = f(\pi) = 2e^\pi - 3\underbrace{\sin \pi}_{=0} + C(\pi) + \underbrace{D}_{=-2}$$

$$0 = 2e^\pi + C\pi - 2$$

$$C\pi = 2 - 2e^\pi$$

$$C = \frac{2 - 2e^\pi}{\pi}$$

$$\boxed{f(t) = 2e^t - 3\sin t + \left(\frac{2 - 2e^\pi}{\pi}\right)t - 2}$$

§4.2 Verify that the function satisfies the hypotheses of Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

#12 $f(x) = x^3 + x - 1$, $[0, 2]$

- f is continuous on $[0,2]$
  f' exists on $(0,2)$ (that is, f is differentiable) because f is a polynomial.
- MVT $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ Find c $a=0$, $b=2$ $f(x) = x^3 + x - 1$

LHS $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

RHS $\dfrac{f(b) - f(a)}{b-a} = \dfrac{f(2) - f(0)}{2-0} = \dfrac{[(2)^3 + 2 - 1] - [0^3 + 0 - 1]}{2}$

$= 5$

LHS = RHS

$3x^2 + 1 = 5$

$3x^2 = 4$

$x^2 = \dfrac{4}{3}$

$x = \pm\sqrt{\dfrac{4}{3}} = \pm\dfrac{2}{\sqrt{3}} = \pm\dfrac{2\sqrt{3}}{3}$

negative not in $[0, 2]$.

$\boxed{c = \dfrac{2\sqrt{3}}{3}}$

§4.4 Find the limit.

#61 $\lim_{x \to 0^+} (4x+1)^{\cot x}$ $\quad 1^\infty$

$$= \lim_{x \to 0^+} e^{\boxed{\ln (4x+1)^{\cot x}}}$$

$$\lim_{x \to 0^+} \ln (4x+1)^{\cot x}$$

$$= \lim_{x \to 0^+} \cot x \cdot \ln (4x+1) \quad \to \ln 1 = 0$$

$$= \lim_{x \to 0^+} \frac{\ln (4x+1)}{\tan x} \quad \to 0 \qquad \frac{0}{0} \text{ form}$$

$$\overset{LH}{=} \lim_{x \to 0^+} \frac{\left(\frac{4}{4x+1}\right)}{\sec^2 x} = \frac{\frac{4}{4(0)+1}}{\underset{=1}{\sec^2(0)}} = 4$$

Final Answer: e^4

§4.1 Find the absolute max & min on the given interval.

#54 ~~#55~~

$$f(x) = \frac{x^2-4}{x^2+4}, \quad [-4, 4]$$

<u>SOLUTION</u> Critnums.

$$f'(x) = \frac{(x^2-4)'(x^2+4) - (x^2-4)(x^2+4)'}{(x^2+4)^2}$$

$$= \frac{2x(x^2+4) - (x^2-4)(2x)}{(x^2+4)^2}$$

$$= \frac{\cancel{2x^3} + 8x - \cancel{2x^3} + 8x}{(x^2+4)^2} = 0$$

$$\frac{16x}{(x^2+4)^2} = 0$$

$$\boxed{x = 0}$$

Plot Points

endpt $x = -4$ $f(-4) = \frac{(-4)^2-4}{(-4)^2+4} = \frac{12}{20} = \frac{3}{5}$

critnum $x = 0$ $f(0) = \frac{0^2-4}{0^2+4} = \frac{-4}{4} = -1$

endpt $x = 4$ $f(4) = \frac{(4)^2-4}{(4)^2+4} = \frac{12}{20} = \frac{3}{5}$

Min $(0, -1)$ Max $(\pm 4, \frac{3}{5})$

§4.3 Let $f(x) = 8x^2 - x^4$ ← even.

(i) Find the critical numbers.

$$f'(x) = 16x - 4x^3 = 4x(4 - x^2)$$
$$= 4x(2-x)(2+x), \quad x=0,\ x=\pm 2$$

(ii) Make a chart showing where f is decreasing or increasing.

x	$f'(x)$
3	neg
1	

		$x=-2$		$x=0$		$x=2$	
f	↗	max	↘	min	↗	max	↘
f'	pos	-2	neg	0	pos	2	neg

(iii) Find the inflection points.

$$f''(x) = 16 - 12x^2$$
$$= 4(4 - 3x^2),$$

$4 - 3x^2 = 0$

$3x^2 = 4$

$x^2 = \frac{4}{3}$, $x = \pm\sqrt{\frac{4}{3}}$

$\boxed{x = \frac{\pm 2\sqrt{3}}{3}}$

(iv) Make a chart showing where f is concave up or down.

x	$f''(x)$
10	neg
0	16 pos
-10	neg

f	⌢		⌣		⌢
f''	neg	$-2\sqrt{3}/3$	pos	$2\sqrt{3}/3$	neg

(v) Plot the max's, mins, and inflection points.

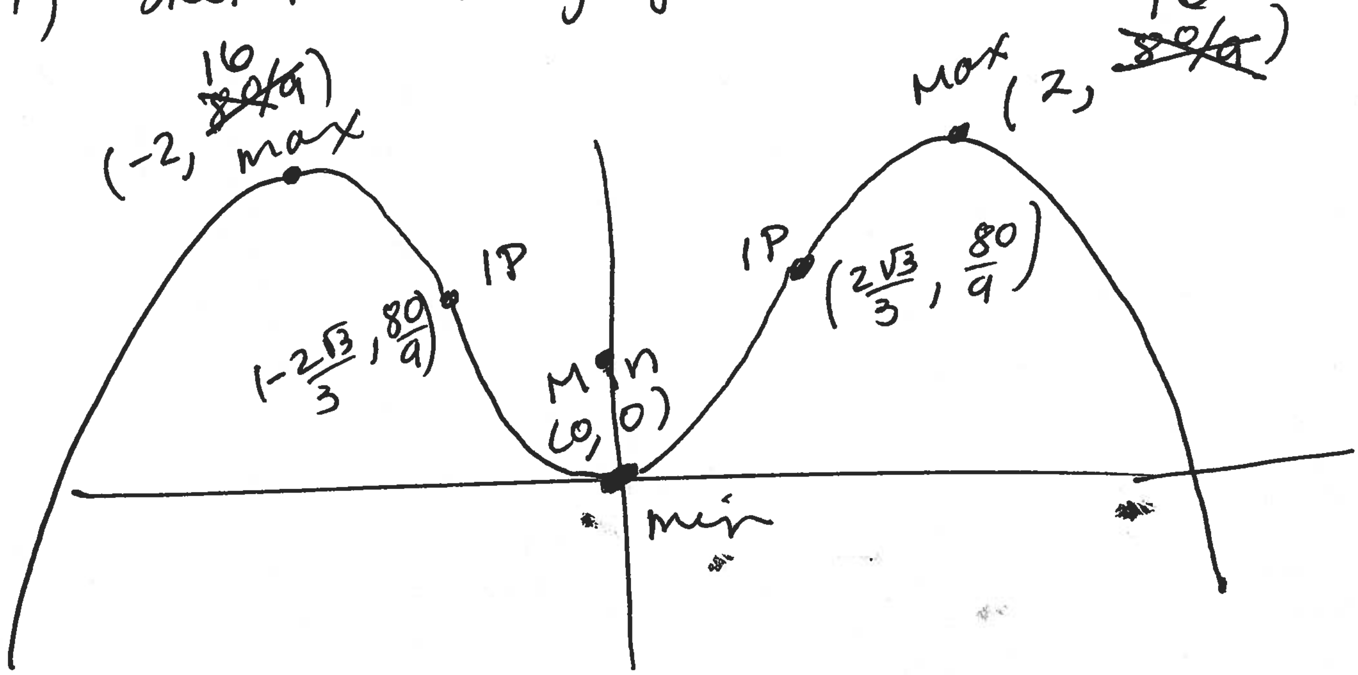
$$f(-2) = 8(-2)^2 - (-2)^4 = 16$$

$$f\left(-\frac{2\sqrt{3}}{3}\right) = 8\left(\frac{-2\sqrt{3}}{3}\right)^2 - \left(\frac{-2\sqrt{3}}{3}\right)^4 = 8\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2$$
$$= \frac{32}{3} - \frac{16}{9} = +\frac{80}{9}$$

$$f\left(\frac{2\sqrt{3}}{3}\right) = +80/9$$

$$f(2) = 16$$

(vi) Sketch the graph



§4.3 Let $f(x) = 2x^5 - 5x^2 + 1$

(i) Find the critical numbers

$$f'(x) = 10x^4 - 10x = 10x(x^3 - 1)$$

$$x = 0, \quad x = 1$$

(ii) Make a chart showing where f is increasing or decreasing.

f: ↗ max $x=0$ ↘ min $x=1$ ↗

f': pos | neg | pos

(0, 1)

x	$f'(x)$
-1	20
$1/2$	$-35/8$ neg
2	140 pos

(iii) Find the inflection points.

$$f''(x) = 40x^3 - 10$$
$$= 10(4x^3 - 1) = 0$$

$$4x^3 - 1 = 0$$
$$x^3 = \frac{1}{4}$$
$$\boxed{x = \frac{1}{4^{1/3}}}$$

(iv) Make a chart showing where f is concave up or down.

f: ⌒ | ◡

f'': neg | pos

$x = 4^{1/3}$

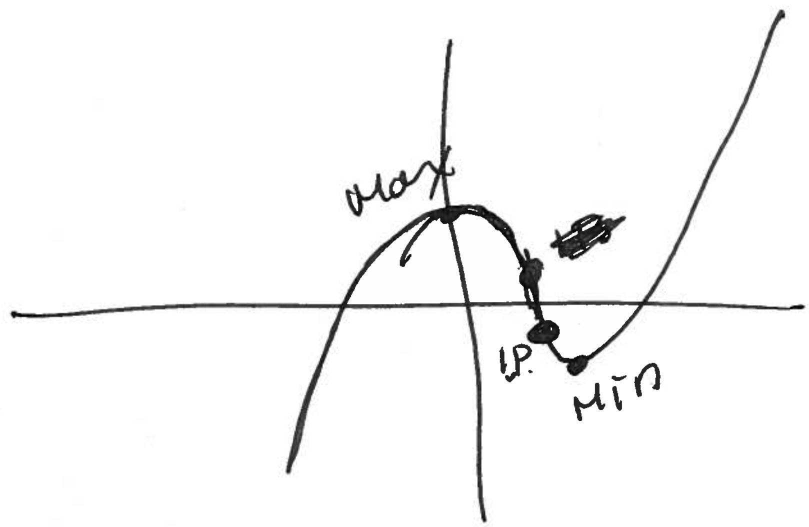
(v) Plot the max's, min's, and I.P.'s

<u>max</u> $x = 0$ $f(0) = 1$

<u>I.P.</u> $x = \frac{1}{\sqrt[3]{4}}$ $f\left(\frac{1}{\sqrt[3]{4}}\right) \approx -.79$

<u>Min</u> $x = 1$ $f(1) = 2(1)^5 - 5(1)^2 + 1 = -2$

(vi) Sketch the graph.



§4.7 #3 Find two positive numbers whose product is 100 and whose sum is a minimum.

FORMULAS

$xy = 100$

$S = x + y$

Eliminate Variable

$y = \frac{100}{x}$

$S = x + \frac{100}{x}$, $x > 0$

$S = x + 100x^{-1}$

Critnums

$S' = 1 + 100(-1 \cdot x^{-2})$

$= 1 - \frac{100}{x^2} = 0$

$1 - \frac{100}{x^2} = 0$

$\frac{100}{x^2} = 1$

$x^2 = 100$

$x = \pm 10$

$\cancel{x = 10}$

~~MAX~~

$\boxed{x = 10 \quad y = \frac{100}{x} = \frac{100}{10} = 10}$

~~MIN~~

$\cancel{x = -10 \quad y = \frac{100}{-10} = -10 \quad S = -20}$

Second Deriv Test

$S'' = 0 + 100(-1)(-2)x^{-3}$

$= \frac{200}{x^3}$

$S''(10) = \frac{200}{(10)^3}$ = pos ∪

$x = 10$ ~~max~~ min

$S''(-10) = \frac{200}{(-10)^3}$ neg

$x = -10$ max

§4.2 Verify that the function satisfies the hypothesis of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

#13 $f(x) = e^{-2x}$, $[0, 3]$

- f is continuous on $[0,3]$ and ~~differentiable~~ f' exists on $(0,3)$ because f is an exponential function.

- MVT

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$a=0$,
$b=3$,
$f(x) = e^{-2x}$

Find c

LHS

$$f(x) = e^{(-2x)}$$

$$f'(x) = -2e^{-2x}$$

RHS

$$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(0)}{3-0} = \frac{e^{-2(3)} - e^{-2(0)}}{3}$$

$$= \frac{e^{-6}-1}{3}$$

LHS = RHS

$$-2e^{-2x} = \frac{e^{-6}-1}{3}$$

$$e^{-2x} = -\frac{1}{2}\left(\frac{e^{-6}-1}{3}\right)$$

$$e^{-2x} = \frac{1-e^{-6}}{6}$$

$$\ln e^{-2x} = \ln\left(\frac{1-e^{-6}}{6}\right)$$

$$-2x = \ln\left(\frac{1-e^{-6}}{6}\right)$$

$$\boxed{x = -\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right)}$$

§4.1 #60 $f(x) = x - \ln x$, $[\frac{1}{2}, 2]$

Find the absolute max & min.

Critnums

$$f'(x) = 1 - \frac{1}{x}$$

$$1 - \frac{1}{x} = 0$$

$$\frac{x}{x} - \frac{1}{x} = 0$$

$$\frac{x-1}{x} = 0$$

$x - 1 = 0$, $\boxed{x = 1}$

Plot Points

endpoints $x = \frac{1}{2}$, $f(\frac{1}{2}) = \frac{1}{2} - \ln(\frac{1}{2}) \approx 1.19$

critnums $x = 1$, $f(1) = 1 - \ln 1 = 1 - 0 = 1$

endpt $x = 2$, $f(2) = 2 - \ln 2 \approx 1.3$

$$\boxed{\begin{array}{l} \text{Abs Min } (1, 1) \\ \text{Abs Max } (2, 2 - \ln 2) \end{array}}$$

§4.1 Find the absolute max & min of f on the interval.

#53 $f(x) = \dfrac{x}{x^2+1}$, $[0, 2]$

Critnums

$$f'(x) = \frac{(x)'(x^2+1) - (x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(1)(x^2+1) - (x)(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1 - 2x^2}{(x^2+1)^2}$$

$$= \frac{-x^2+1}{(x^2+1)^2} = 0$$

$-x^2+1=0$

$x^2=1$, $x = \pm 1$

$\boxed{x=1}$ is in $[0, 2]$

Plot Points

endpt

$f(0) = \dfrac{0}{0^2+1} = 0$

critnum

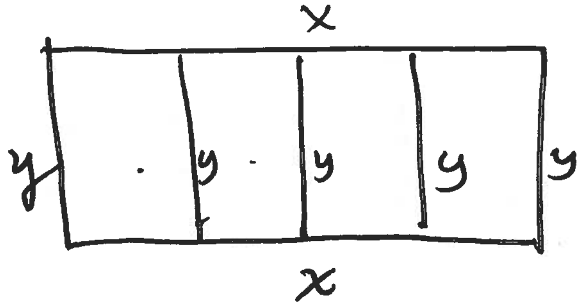
$f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$

endpt

$f(2) = \dfrac{2}{2^2+1} = \dfrac{2}{5}$

Abs Min $(0, 0)$
Abs Max $(1, \frac{1}{2})$

§4.7#9 A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible area of the ~~four~~ pens?



<u>FORMULAS</u>

$A = xy$

$2x + 5y = 750$

<u>Eliminate Variable</u>

$2x + 5y = 750$

$5y = -2x + 750$

$y = \frac{-2}{5}x + \frac{750}{5}$

$y = \frac{-2}{5}x + 150$

$A = x\left(\frac{-2}{5}x + 150\right)$

$A = \frac{-2}{5}x^2 + 150x$

<u>Critnums</u>

$A' = \frac{-2}{5}(2x) + 150$

$= -\frac{4}{5}x + 150 = 0$

$\frac{4}{5}x = 150$

$x = \frac{5}{4}(150)$

$x = \frac{750}{4} = \frac{375}{2}$

$x = 187.5$ ft

<u>Find the area</u>

Area $A = \frac{-2}{5}x^2 + 150x$

$= \frac{-2}{5}\left(\frac{375}{2}\right)^2 + 150\left(\frac{375}{2}\right)$

$= 14{,}062.5$ sq. ft.

§4.1 # 54 Find the absolute max & min.

$$f(x) = \frac{x^2-4}{x^2+4}, \quad [-4, 4]$$

SOLUTION

Critnums

$$f'(x) = \frac{(x^2-4)'(x^2+4) - (x^2-4)(x^2+4)'}{(x^2+4)^2}$$

$$= \frac{2x(x^2+4) - (x^2-4)(2x)}{(x^2+4)^2}$$

$$= \frac{2x^3+8x-2x^3+8x}{(x^2+4)^2}$$

$$= \frac{16x}{(x^2+4)^2} = 0$$

$$\boxed{x=0}$$

Plot Points

$$f(-4) = \frac{(-4)^2-4}{(-4)^2+4} = \frac{16-4}{16+4} = \frac{12}{20} = \frac{3}{5}$$

$$f(0) = \frac{0^2-4}{0^2+4} = -1$$

$$f(4) = \frac{3}{5}$$

Abs Max $(\pm 4, \frac{3}{5})$

Abs Min. $(0, -1)$

§4.1 #59 Find the absolute max & min.

$$f(x) = x e^{-x^2/8}, \quad [-1, 4]$$

SOLUTION Critnums

$$f'(x) = (x)'(e^{-x^2/8}) + (x)(e^{-x^2/8})'$$

$$f'(x) = 1 \cdot e^{-x^2/8} + x(e^{-x^2/8})\left(-\frac{2x}{8}\right) = 0$$

$$e^{-x^2/8}\left(1 - \frac{x^2}{4}\right) = 0$$

$$1 - \frac{x^2}{4} = 0$$

$$\frac{x^2}{4} = 1$$

$$x^2 = 4$$

$$x = \pm 2$$

$\boxed{x = 2}$ is in $[-1, 4]$

Evaluate

$$f(-1) = (-1)e^{-(-1)^2/8} = -e^{-1/8}$$

$$f(2) = (2)e^{-(2)^2/8} = 2e^{-4/8} = 2e^{-1/2} = \frac{2}{\sqrt{e}} \approx 1.21$$

$$f(4) = 4e^{-(4)^2/8} = 4e^{-2} = \frac{4}{e^2} \approx .54$$

Min: $(-1, -e^{-1/8})$

Max: $\left(2, \frac{2}{\sqrt{e}}\right)$

§4.2 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. The find all numbers c that satisfy the conclusion of the Mean Value Theorem.

#14 $f(x) = \dfrac{x}{x+2}$, $[1, 4]$

SOLUTION

- f is contin on $[1, 4]$ and ~~differentiable~~ f' exists on $(1, 4)$ because it is a rational function
- Mean Value Theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Find c. $a = 1$, $b = 4$

$f(x) = \dfrac{x}{x+2}$

RHS $\dfrac{f(b) - f(a)}{b - a} = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{\left(\frac{4}{4+2}\right) - \left(\frac{1}{1+2}\right)}{3}$

$= \dfrac{\left(\frac{2}{3} - \frac{1}{3}\right)}{3} = \dfrac{\left(\frac{1}{3}\right)}{3}$

$= \dfrac{1}{9}$

LHS: $f'(x) = \dfrac{(x)'(x+2) - (x)(x+2)'}{(x+2)^2}$

$= \dfrac{(x+2) - (x)}{(x+2)^2} = \dfrac{2}{(x+2)^2}$

LHS = RHS $\dfrac{2}{(x+2)^2} = \dfrac{1}{9}$

$$(x+2)^2 = 18$$

$$x+2 = \pm\sqrt{18}$$

$$x = -2 \pm \sqrt{9}\sqrt{2}$$

$$x = -2 \pm 3\sqrt{2}$$

only $\boxed{x = -2+3\sqrt{2}}$ lies in $[1,4]$

$$\boxed{c = -2+3\sqrt{2}}$$

§4.3 Let $f(x) = x^4 - 4x^3$

(i) Find the critical numbers

$$f'(x) = 4x^3 - 12x^2$$
$$4x^2(x-3) = 0$$
$$x = 0, \quad x = 3$$

(ii) Make a chart showing where f is increasing or decreasing

f: ↘ neither ↘ min ↗

f': neg | neg | pos

−1 0 3

x	$f'(x)$
4	64
1	−8
−1	−16

(iii) Find the inflection points

$$f''(x) = 12x^2 - 24x = 0$$
$$12x(x-2) = 0, \qquad x = 0, \ x = 2$$

(iv) Make a chart showing where f is concave up or down.

f: conc. up | $x=0$ I.P. | down | $x=2$ I.P. | up

f'': pos | neg | pos

−1 0 1 2 3

x	$f''(x)$
3	36
1	−12
−1	36

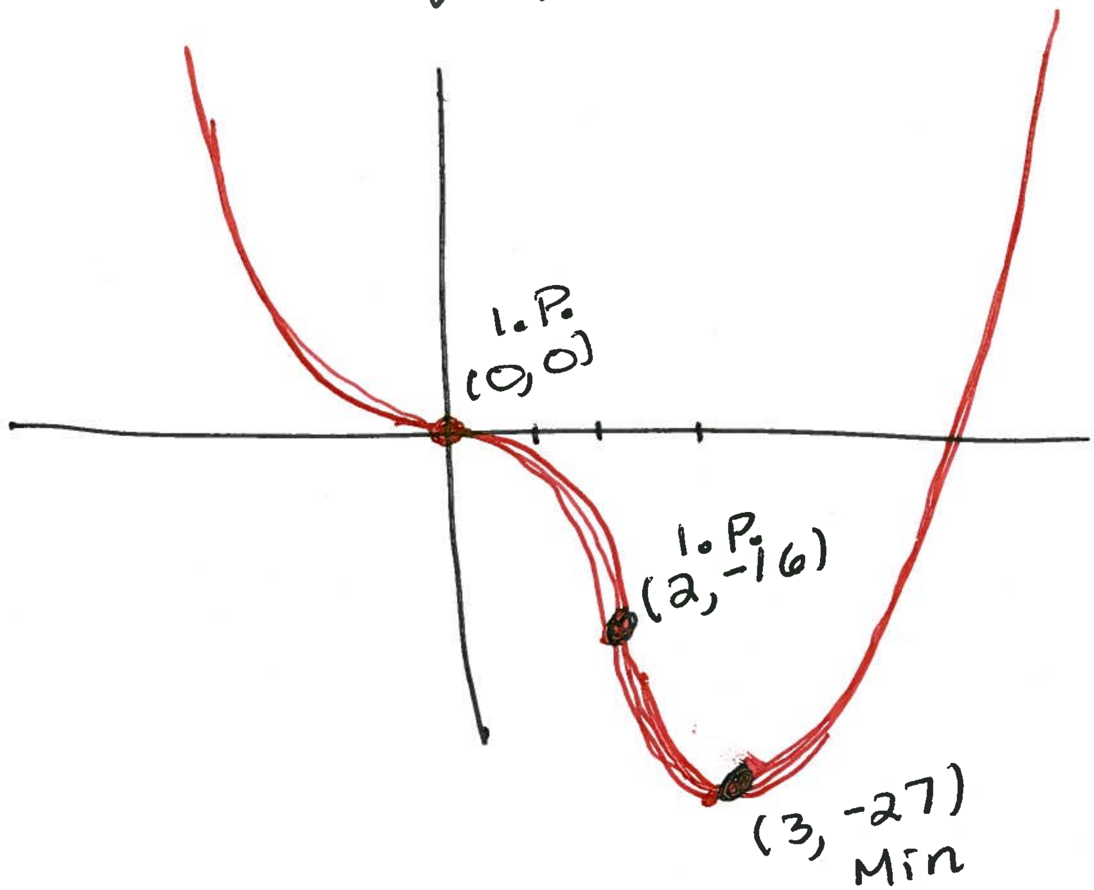
(v) Plot the mins, max's and inflection points

I.P. $x=0$ $f(0) = 0$

I.P. $x=2$ $f(2) = (2)^4 - 4(2)^3 = -16$

Min $x=3$ $f(3) = (3)^4 - 4(3)^3 = -27$

(vi) Sketch the graph.



§4.9 Find f #39

$$f''(\theta) = \sin\theta + \cos\theta, \quad f(0)=3, \quad f'(0)=4$$

$$f'(\theta) = -\cos\theta + \sin\theta + C$$

$$4 = f'(0) = -\cos(0) + \sin(0) + C$$

$$4 = -1 + 0 + C$$

$$\boxed{C=5}$$

$$f'(\theta) = -\cos\theta + \sin\theta + 5$$

$$f(\theta) = -\sin\theta + -\cos\theta + 5\theta + D$$

$$3 = f(0) = -\sin(0) - \cos(0) + 5(0) + D$$

$$3 = -0 - 1 + 0 + D$$

$$3 = -1 + D$$

$$\boxed{D=4}$$

$$\boxed{f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4}$$