4.6 L'Hospital's Rule

We return now to the problem of finding limits of functions. For continuous functions, we can find the limit by direct substitution. Let's consider the limit

$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$

Direct substitution gives a fraction $\frac{0}{0}$. In the case of certain algebraic functions, we could simplify and cancel out a factor which would allow us to take the limit by direct substitution. This is not possible for the limit shown above. We have the following important theorem which we can use to find the above limit.

Theorem 4.1 L'Hospital's Rule Suppose that f and g are differentiable and $g'(x) \neq 0$ near a, although possibly g is zero at a. Moreover, the number a can be replaced by $\pm\infty$. Suppose also that

$$\begin{array}{ll} \lim_{x\to a} f(x) = 0 & \text{and} \quad \lim_{x\to a} g(x) = 0 \\ \text{or that} & \\ \lim_{x\to a} f(x) = \pm\infty & \text{and} \quad \lim_{x\to a} g(x) = \pm\infty \end{array}$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the limit on the right hand side exists.

Proof Let's do the proof of the special case when $g'(x) \neq 0$ and the indeterminate form is $\frac{0}{0}$.

$$\begin{aligned} \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} &= \frac{\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a}} \\ &= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \\ &= \lim_{x\to a} \frac{f(x)}{g(x)} \end{aligned}$$

Example 1 Find $\lim_{x\to 1} \frac{\ln x}{x-1}$.
SOLUTION The limit is of a form $\frac{0}{0}$.

$$\lim_{x\to 1} \frac{\ln x}{x-1} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = 0 \quad \square$$

Example 2 Find $\lim_{x\to\infty} \frac{e^x}{x^2}$.
SOLUTION The limit is of a form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'H}}{=} \frac{e^x}{2x} \overset{\text{L'H}}{=} \frac{e^x}{2} = \infty \quad \square$$

Example 3 Find $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.
SOLUTION The limit is of a form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/3}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} 3x^{-1}x^{2/3} = \lim_{x\to\infty} 3x^{-1/3} = \lim_{x\to\infty} \frac{3}{x^{1/3}} = 0 \quad \square$$

Example 4 Find $\lim_{x\to\pi} \frac{\sin x}{1+\cos x}$.
The limit is of a form $\frac{0}{0}$.

$$\lim_{x\to\pi} \frac{\sin x}{1+\cos x} \overset{\text{L'H}}{=} \lim_{x\to\pi} \frac{\cos x}{1-\sin x} = \frac{\cos\pi}{1-\sin\pi} = \frac{-1}{1-0} = -1 \quad \square$$

Indeterminate Products

This is a limit of the form $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or the other way around, where $f \to \infty$ and $g \to 0$). This is called an indeterminate product of the form $0 \cdot \infty$ because we cannot tell right away whether the product should be very large, ∞, or very small, 0, or somewhere in between.

We will use a technique to rewrite the product $f(x)g(x)$ into a fraction. We then can apply L'Hospital's Rule.

$$\lim_{x\to a} f(x)g(x) = \lim_{\to a} \frac{f(x)}{1/g(x)},$$

or

$$\lim_{x\to a} f(x)g(x) = \lim_{\to a} \frac{g(x)}{1/f(x)},$$

Example 5 Find $\lim_{x\to 0^+} x\ln x$.

Solution This limit is of the form $0\cdot\infty$.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{\text{L'H}}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0 \qquad \square$$

Indeterminate Differences

We now study limits of the form $\lim_{x\to a}(f(x)-g(x))$ that are of the form $\infty-\infty$. Throughout this section, we should keep in mind that ∞ is not a number. The form $\infty-\infty$ means that we have two functions that are both getting large, and we are subtracting the two values. The answer should not necessarily be 0.

For indeterminate differences, we try to write the difference of the two functions as a fraction so that we can apply L'Hospital's Rule, if possible.

Example 6 Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

Solution This limit is of the form $\infty-\infty$.

$$\lim_{x\to(\pi/2)^-}(\sec x-\tan x) = \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right) = \lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x}$$

This limit has indeterminate form $0/0$, so we can apply L'Hospital's Rule.

$$\lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x} \overset{\text{L'H}}{=} \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = \frac{0}{1} = 0 \qquad \square$$

Indeterminate Powers

Consider the limit of an expression of the form

$$\lim_{x\to a}[f(x)]^{g(x)}.$$

There are three types of indeterminate forms relating to this expression. By using some algebraic manipulation, the limit can be written in a form where L'Hospital's Rule can be applied.

- type 0^0.
- type ∞^0
- type 1^∞

We employ a technique that uses the cancelation property of logarithms $e^{\ln a} = a$. We also use the property of logarithms $\ln A^x = x \ln A$.

Example 7 Calculate $\lim_{x\to 0}(1+4x)^{3/x}$.
SOLUTION This limit is of the form 1^∞ which is an indeterminate power.

STEP 1: Use the cancelation property of logarithms and simplify the exponent.

$$\lim_{x\to 0}(1+4x)^{3/x} = \lim_{x\to 0} e^{\ln(1+4x)^{3/x}} = \lim_{x\to 0} e^{\frac{3}{x}\cdot\ln(1+4x)}$$

STEP 2: Evaluate the limit of the function that is in the exponent. We will find the limit, call it L, and then our final answer will be e^L.

$$\lim_{x\to 0}\frac{3}{x}\cdot\ln(1+4x) = \lim_{x\to 0}\frac{3\ln(1+4x)}{x}$$

This limit has indeterminate form $\frac{0}{0}$, so we can use L'Hospital's Rule.

$$\lim_{x\to 0}\frac{3\ln(1+4x)}{x} \stackrel{\text{L'H}}{=} \lim_{x\to 0}\frac{3\frac{1}{(1+4x)}\cdot 4}{1} = \frac{3\frac{1}{(1+40)}\cdot 4}{1} = 12$$

STEP 3: Get the final answer by substituting back the limit of the exponent.

$$\lim_{x\to 0}(1+4x)^{3/x} = e^{12} \qquad \square$$