4.5 Applied Optimization Problems

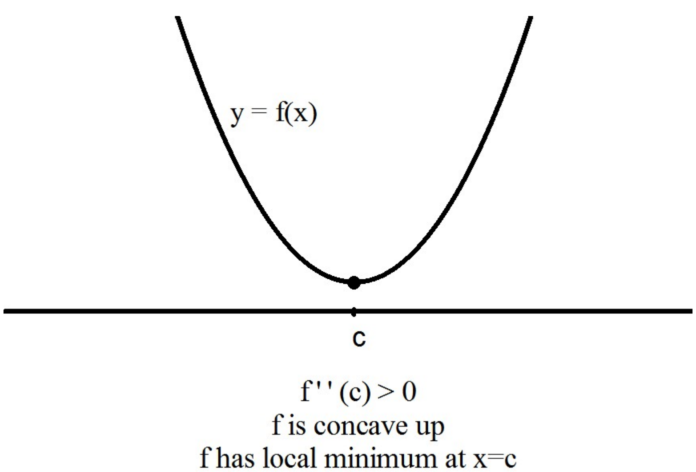
In this section, we find the maximum or minimum values of functions that are from applied problems. We first present a method to determine whether a critical number of a function is either a local maximum, local minimum, or neither.

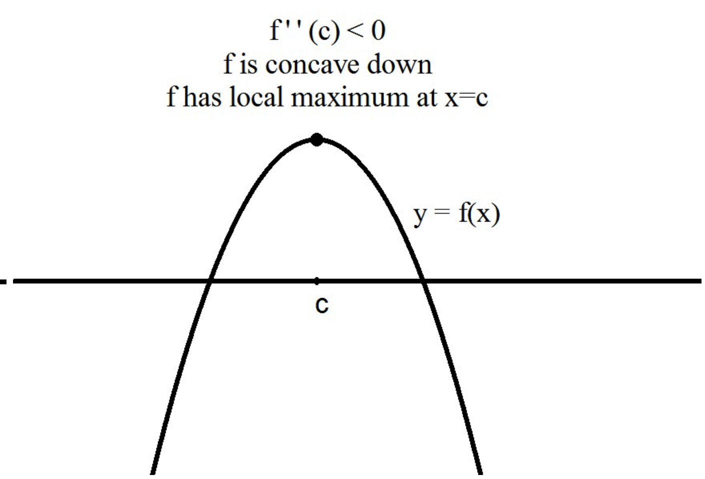
To determine whether a critical number $x = c$ of a function f is a local maximum or a local minimum, we can make a chart which shows where the function f is increasing or decreasing.

Another way to determine whether a critical number $x = c$ of a function f is a local maximum or local minimum is to find the second derivative of the function. If we evaluate the second derivative f'' at the critical number, we can determine whether the function function f is concave up or down around the critical number.

If the second derivative is positive around the critical number, $f''(c) > 0$, then the function is concave up near $x = c$, and therefore, f has a local minimum at $x = c$.

If the second derivative is negative around the critical number, $f''(c) < 0$, then the function is concave down near $x = c$, and therefore, f has a local maximum at $x = c$.





The Second Derivative Test Suppose that f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has local minimum at c.
2. If $f(c) > 0$ and $f''(c) < 0$, then f has local maximum at c.

We now present worked examples of applied optimization problems.

Example 1 Suppose that $x + y = 100$. Find values for x and y such that the product xy is greatest.

SOLUTION

Formulas

$$\begin{array}{ll} \text{Product:} & P = xy \\ \text{Sum:} & x + y = 100 \end{array}$$

Find the values for x and y when P is a maximum.

Eliminate One of the Variables

$$\begin{aligned} y &= -x + 100 \\ P &= xy \\ &= x(-x + 100) \\ &= -x^2 + 100x, \qquad 0 \le x \le 100 \end{aligned}$$

Critical Numbers

$$P' = -2x + 100 = 0$$

$$x = 50$$

Check that Critical Number Represents a Maximum

$$\begin{aligned} P'(x) &= -2x + 100 \\ P''(x) &= -2 \\ P''(50) &= -2 < 0 \end{aligned}$$

We see that P'' is negative at $x = 50$. This means that the function P is concave down around $x = 50$, and therefore P has a maximum value at $x = 50$.

Another way to see that $x = 50$ must represent a maximum is to note that $P = -x^2 + 100x$ is a downward facing parabola. The critical point is the vertex of the parabola and is therefore a maximum value of the function.

Solve for the Other Variable

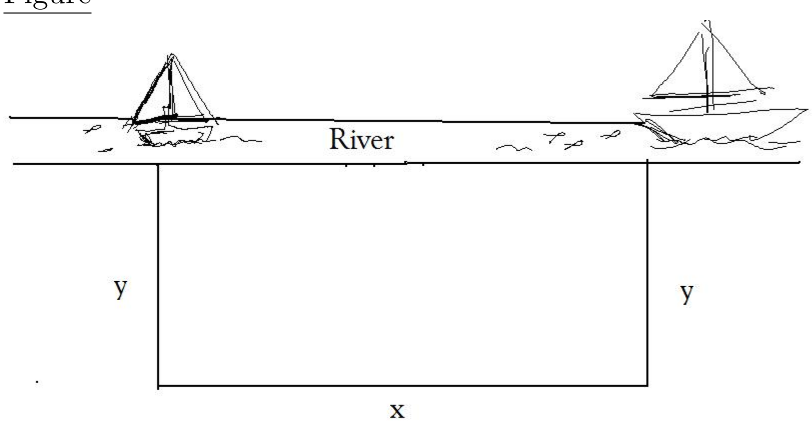
$$\begin{aligned} y &= -x + 100 \\ &= -(50) + 100 = 100 \end{aligned}$$

Final Answer: $x = 50$ and $y = 50$. □

Example 2 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.

Solution

Figure



Formulas

$$\begin{array}{ll} \text{Area:} & A = xy \\ \text{Fencing:} & x + 2y = 2400 \end{array}$$

Find the values for x and y when area is a maximum.

Eliminate One of the Variables

$$\begin{aligned} x &= -2y + 2400 \\ A &= (-2y + 2400)y \\ A &= -2y^2 + 2400y, \quad 0 \leq y \leq 1200 \end{aligned}$$

Critical Numbers

$$A' = -4y + 2400 = 0$$

$$y = 600 \text{ ft}$$

Check that Critical Number Represents a Maximum

$$\begin{aligned} A' &= -4y + 2400 \\ A'' &= -4 \\ A'' &= -4 < 0 \end{aligned}$$

We see that A'' is negative at $y = 600$. This means that the function A is concave down around $y = 600$, and therefore A has a maximum value at $y = 600$.

Another way to see that $y = 600$ must represent a maximum is to note that $A = -2y^2 + 2400y$ is a downward facing parabola. The critical point is the vertex of the parabola. It must be the maximum value.

Finally, another way to see that $A(y)$ attains a maximum at $y = 600$ is to use the fact that $A(y)$ is defined on the closed interval $0 \leq y \leq 1200$. To find the absolute maximum values of a function on a closed interval, we evaluate the function at critical numbers and endpoints. We see that $A(600) = 720,000$, $A(0) = 0$, and $A(1200) = 0$. Therefore $y = 600$ is an absolute maximum value.

Solve for the Other Variable

$$x = -2y + 2400$$

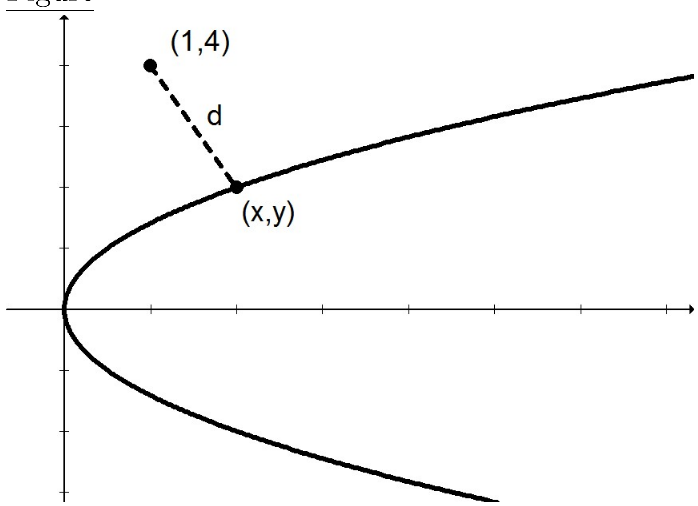
$$x = -2(600) + 2400 = 1200 \text{ ft}$$

Final Answer: The dimensions that give the maximum area are 600 ft deep and 1200 ft wide. □

Example 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

SOLUTION

Figure



Formulas

Distance

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d = \sqrt{(x-1)^2 + (y-4)^2}$$
$$D = d^2 = (x-1)^2 + (y-4)^2$$

Equation of Parabola $x = y^2/2$

We introduce $D = d^2$ because it will be easier to find the derivative of this function. The values of x and y that make D a minimum value are the same values that make d a minimum value.

Find the values for x and y when D is minimum.

Eliminate One of the Variables

$$x = y^2/2$$

$$D = (y^2/2 - 1)^2 + (y - 4)^2$$

Critical Numbers

$$\begin{aligned} D' &= 2(y^2/2 - 1)(y) + 2(y - 4) = 0 \\ y^3 - 2y + 2y - 8 &= 0 \\ y^3 - 8 &= 0 \\ y^3 &= 8 \\ y &= 2 \end{aligned}$$

Check that Critical Number Represents a Minimum

$$\begin{aligned} D'(y) &= y^3 - 8 \\ D''(y) &= 3y^2 \\ D''(2) &= 3(2)^2 > 0 \end{aligned}$$

Around the number $y = 2$, $D''(y)$ is positive, and therefore around $y = 2$ the function $D(y)$ is concave up. This means that $y = 2$ is a minimum value the function $D(y)$.

Solve for the Other Variable

$$x = y^2/2$$

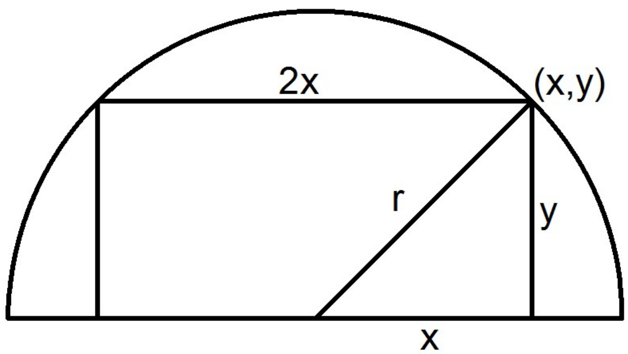
$$x = (2)^2/2 = 2$$

Final Answer: The point on the parabola $2x = y^2$ that is closest to the point $(1, 4)$ is the point $(2, 2)$. $\square$

Example 4 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

SOLUTION

Figure



Formulas

Equation of a Circle/Pythagorean Theorem	$x^2 + y^2 = r^2$
Area of Rectangle	$A = (2x)(y)$

Find the values for x and y when A is maximum.

Eliminate One of the Variables

$$\begin{aligned} A &= 2xy \\ \mathcal{A} &= A^2 = 4x^2y^2 \\ y^2 &= r^2 - x^2 \\ \mathcal{A} &= 4x^2(r^2 - x^2) = 4r^2x^2 - 4x^4, \quad 0 \le x \le r \end{aligned}$$

We introduce $\mathcal{A} = A^2$ because it will be easier to find the derivative of this function. The values of x and y that make $\mathcal{A}$ a minimum value are the same values that make A a minimum value.

Critical Numbers Note that r is a constant throughout this problem. We take the derivative with respect to x.

$$\begin{aligned}\mathcal{A} &= 4r^2x^2 - 4x^4\\ \mathcal{A}' &= 4r^2(2x) - 4(4x^3)\\ &= 8r^2x - 16x^3\\ &= 8x(r^2 - 2x^2) = 0\end{aligned}$$

$$\begin{aligned}x = 0 &\quad \text{or} \quad r^2 - 2x^2 = 0\\ x = 0 &\quad \text{or} \quad x = \sqrt{r^2/2}\\ x = 0 &\quad \text{or} \quad x = \tfrac{r\sqrt{2}}{2}\end{aligned}$$

Note that the length of the rectangle is $2x$, and therefore the area is maximum when the length is

$$2(r\sqrt{2}/2) = r\sqrt{2}$$

<u>Check that Critical Number Represents a Minimum</u>

$$\begin{aligned}\mathcal{A}'(x) &= 8r^2x - 16x^3\\ \mathcal{A}''(x) &= 8r^2 - 48x^2\\ \mathcal{A}''(r\sqrt{2}/2) &= 8r^2 - 48(r\sqrt{2}/2)^2 = 8r^2 - 48(r^2/2) = -16r^2 < 0\end{aligned}$$

Around the number $x = r\sqrt{2}/2$, $\mathcal{A}''(x)$ is negative, and therefore around $x = r\sqrt{2}/2$ the function $\mathcal{A}(x)$ is concave up. This means that $x = r\sqrt{2}/2$ is a minimum value the function $\mathcal{A}(x)$.

<u>Final Answer</u> For this problem, we are not asked to find the dimensions of the inscribed rectangle, so we do not need to solve for the other variable y. We are asked to find the area of the largest rectangle that can be inscribed in the semi-circle. When $x = r\sqrt{2}/2$, the area is

$$\begin{aligned}A &= 2x\sqrt{r^2 - x^2}\\ &= 2\left(r\sqrt{2}/2\right)\sqrt{r^2 - (r\sqrt{2}/2)^2}\\ &= 2\left(r\sqrt{2}/2)\right)\sqrt{r^2 - r^2/2}\\ &= 2\left(r\sqrt{2}/2)\right)\sqrt{r^2/2}\\ &= 2\left(r\sqrt{2}/2)\right)\left(r/\sqrt{2}\right)\\ &= r^2\end{aligned}$$