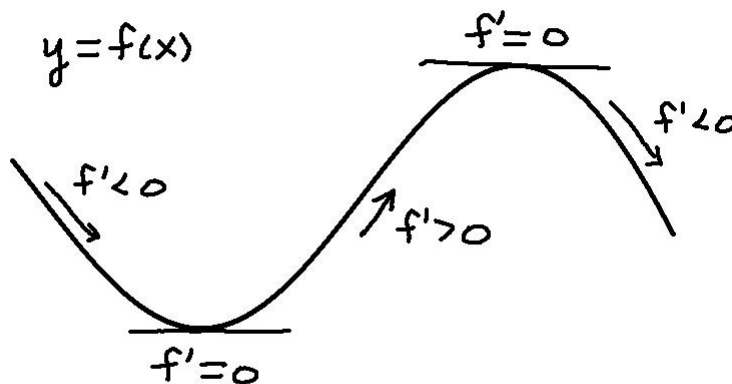


4.3 Derivatives and the Shape of Graphs

The goal of this section is to use the derivative of a function to help us sketch the graph of a function. We could use a graphing calculator to sketch the graph, but we are going to try to sketch the graphs by hand as an exercise and as part of a broader study of the behavior of functions.

The figure below summarizes how derivative and slope of the tangent line are related. When function is increasing when the derivative is positive; the function is decreasing when the derivative is negative. If the derivative exists at a local max or a local min, then the derivative equals zero.



Above we are looking at the graph of a function and drawing conclusions about the derivative. In the next example, we start with a function, and we use derivatives to sketch the graph.

Example 1 Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

1. Find the critical points of the function.

SOLUTION We find $f'(x)$ and then solve $f'(x) = 0$.

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$\begin{aligned}
 f'(x) &= 0 \\
 12x^3 - 12x^2 - 24x &= 0 \\
 12x(x^2 - x - 2) &= 0 \\
 12x(x - 2)(x + 1) &= 0
 \end{aligned}$$

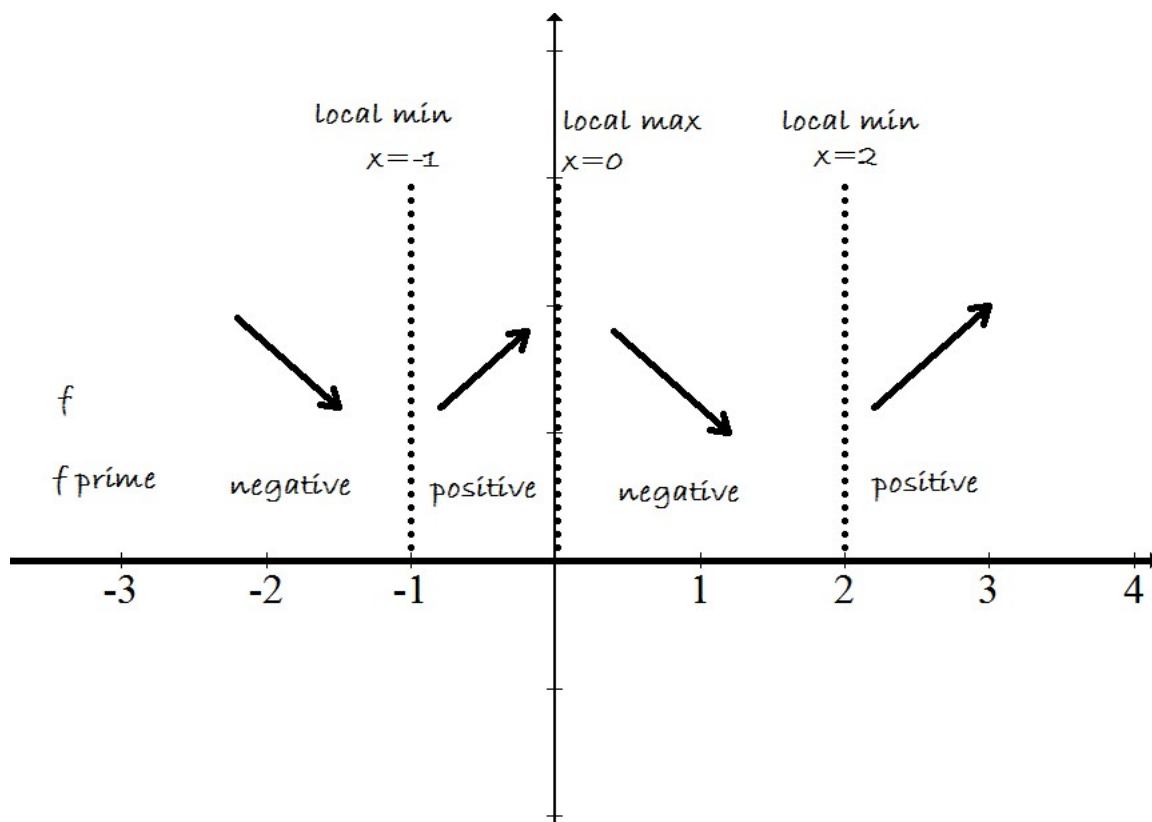
$$x = 0, x = 2, x = -1$$

Just by looking at the critical points alone, there is no way of knowing whether the function has a local max or a local min at these critical points. The next step will tell us whether these points are local maximums or minimums.

2. Make an Increasing/Decreasing Chart. That is, make a chart showing where f' is positive or negative and where f is increasing or decreasing.

SOLUTION

x	$f'(x) = 12x(x - 2)(x + 1)$	
-2	$12(-2)(-4)(-1) = -96$	negative
-1/2	$12(-1/2)(-1/2 - 2)(-1/2 + 1) = 15/2$	positive
1	$12(1)(1 - 2)(1 + 1) = -24$	negative
3	$12(3)(3 - 2)(3 + 1) = 144$	positive



3. Determine the y -coordinates where f has local maximum and local minimum.

SOLUTION We put the x -values into the function f .

$$\text{Local Min } x = -1, \quad f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5 = 0$$

$$\text{Local Max } x = 0, \quad f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 5 = 5$$

$$\text{Local Min } x = 2, \quad f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 5 = -27$$

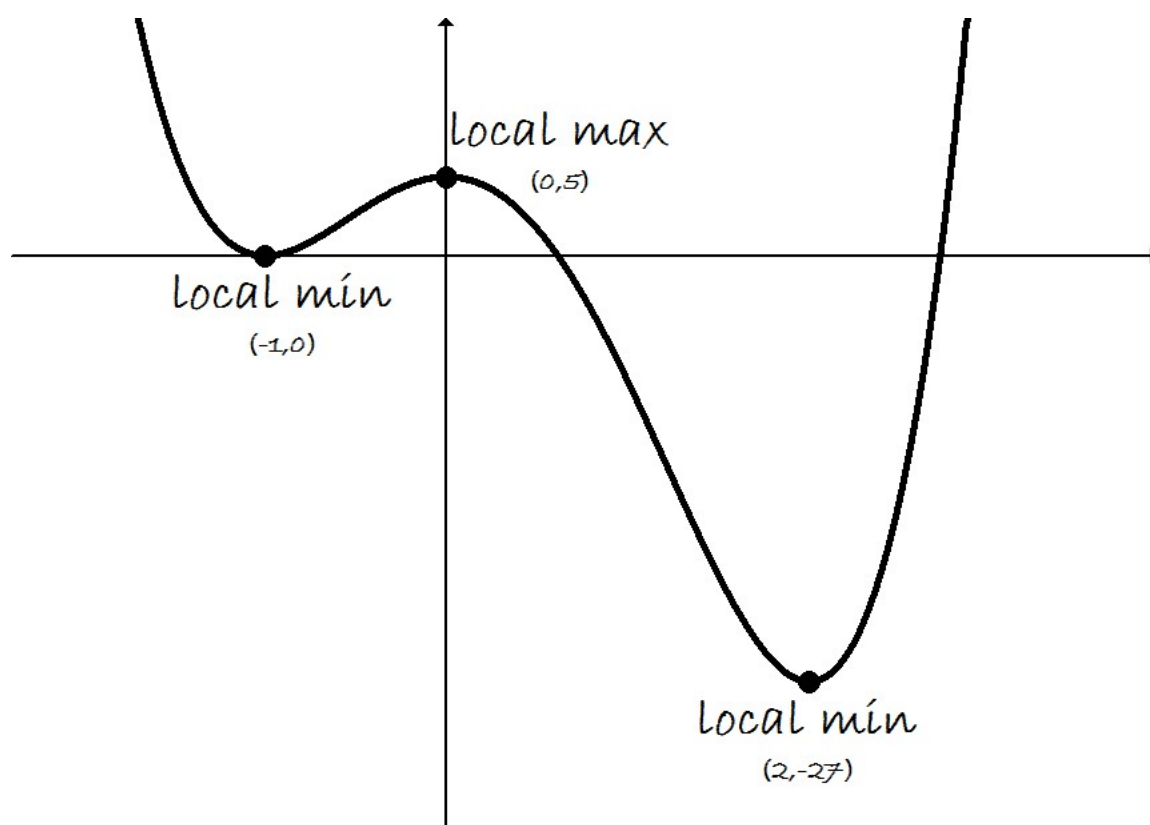
Summarizing, we have

$$\text{Local Min } (-1, 0)$$

$$\text{Local Max } (0, 5)$$

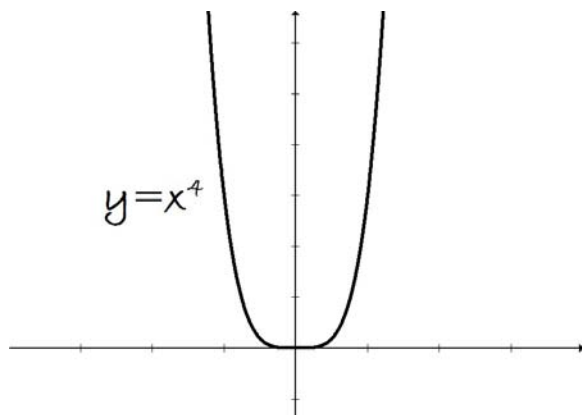
$$\text{Local Min } (2, -27)$$

4. Sketch the graph of f using the above information.



The Behavior of Polynomials as $x \rightarrow \infty$

To draw a complete sketch of the graph, we also need to determine the behavior of the function for large values of x . We will rely on the following principle: *For large values of x , polynomials behave like their highest term.* So for large values of x , the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ behaves like the function $y = 3x^4$. The function $y = 3x^4$ has a graph similar to a the parabola given by $y = x^2$.



We state the limits of the function f as x approaches infinity and negative infinity.

$$\begin{aligned}\lim_{x \rightarrow \infty} (3x^4 - 4x^3 - 12x^2 + 5) &= \lim_{x \rightarrow \infty} 3x^4 = \infty \\ \lim_{x \rightarrow -\infty} (3x^4 - 4x^3 - 12x^2 + 5) &= \lim_{x \rightarrow -\infty} 3x^4 = \infty\end{aligned}$$

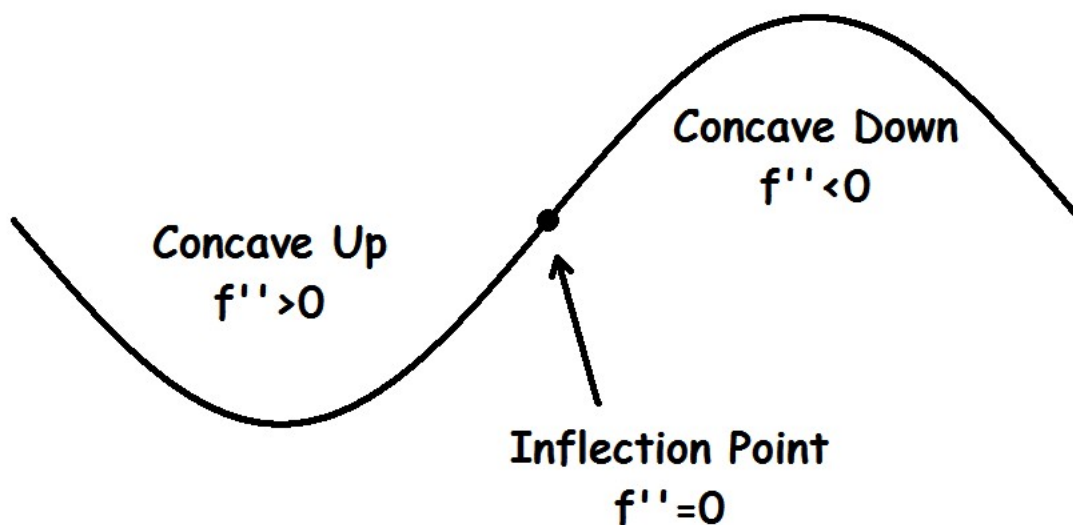
Concavity and the Second Derivative Test

Definition If the graph of f lies above all of its tangent lines on an interval I , then it is called **concave upward** on I . If the graph of f lies below all of its tangent lines on I , then it is called **concave downward** on I .

Concavity Test

1. If $f'' > 0$ for all x in I , then the graph of f is concave upward on I .
2. If $f'' < 0$ for all x in I , then the graph of f is concave downward on I .

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P .



Example 2 Let $f(x) = x^3 - 6x^2 + 9x + 5$.

1. Find the critical points of f .

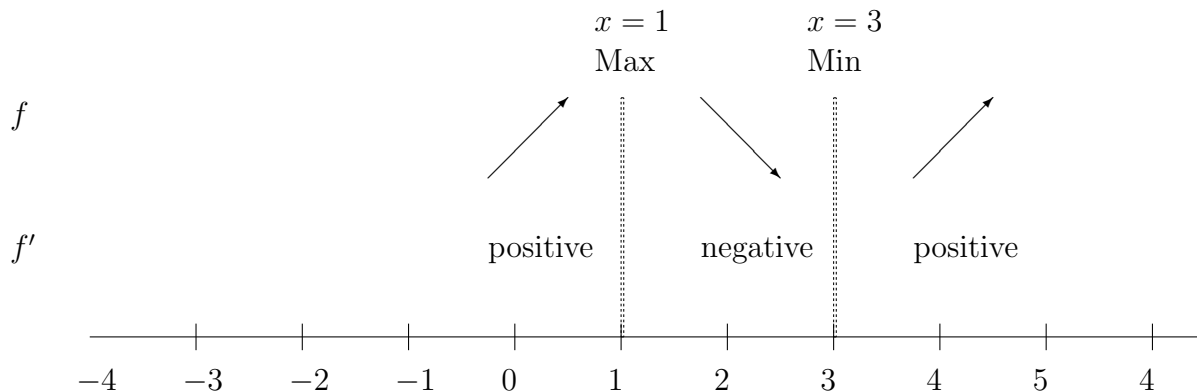
SOLUTION

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 9 \\ 3x^2 - 12x + 9 &= 0 \\ 3(x^2 - 4x + 3) &= 0 \\ 3(x - 1)(x - 3) &= 0 \end{aligned}$$

Critical Points: $x = 1$, $x = 3$

2. Make a chart showing where f is increasing or decreasing.

SOLUTION



x	$f'(x) = 3(x-1)(x-3)$
0	$3(0-1)(0-3) = (-)(-) = +$ positive
2	$3(2-1)(2-3) = (+)(-) = -$ negative
4	$3(4-1)(4-3) = (+)(+) = +$ positive

3. Evaluate

$$\lim_{x \rightarrow \infty} f(x)$$

$$\lim_{x \rightarrow -\infty} f(x)$$

SOLUTION

For large values of x , our function $f(x) = x^3 - 6x^2 + 9x + 5$ behaves like the highest degree term $y = x^3$. We have

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} x^3 = \infty$$

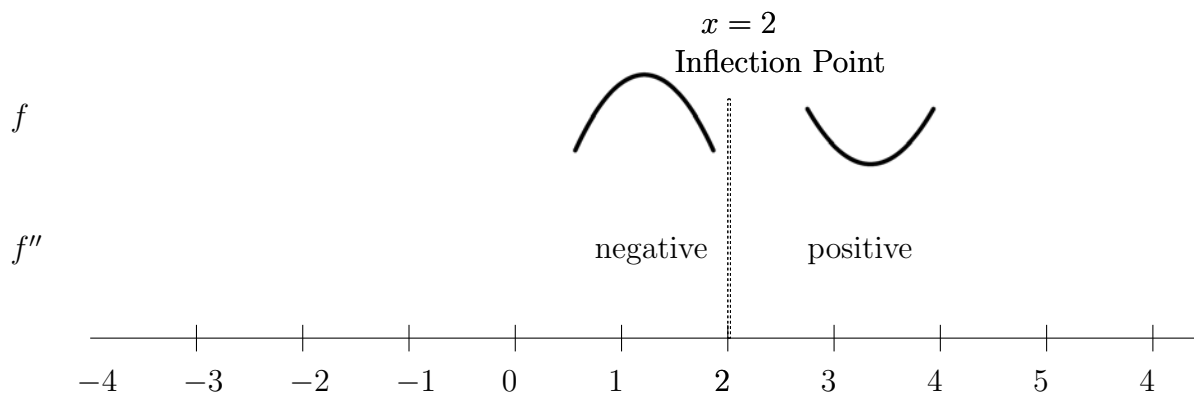
$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} x^3 = -\infty$$

4. Find the possible inflection points of f .

SOLUTION

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 9 \\ f''(x) &= 6x - 12 \\ 6x - 12 &= 0 \\ x &= 2 \end{aligned}$$

5. Make a chart showing where f is concave up and concave down.

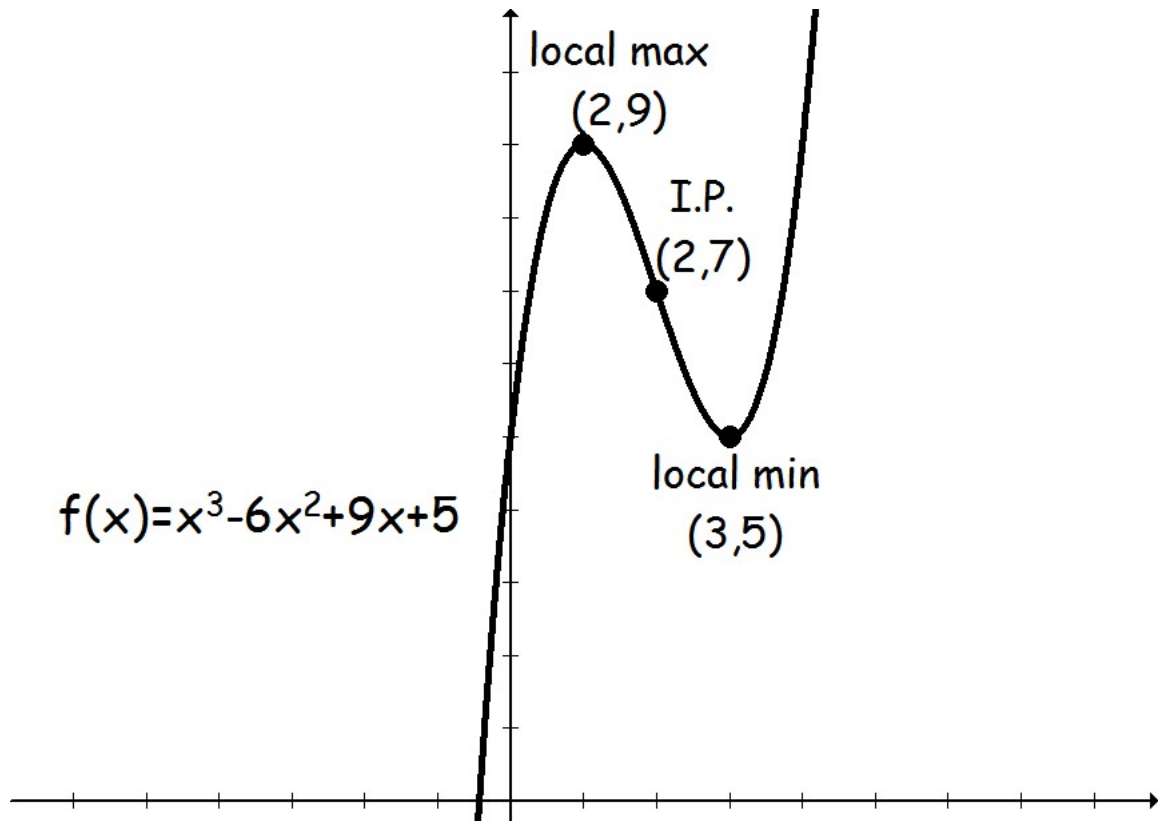


6. Plot Points: Find the y -coordinates for the local maximum, local minimum, and the inflection point.

SOLUTION

	x	$f(x)$
local max	1	$f(1) = 9$
inflection point	2	$f(2) = 7$
local min	3	$f(3) = 5$

7. Sketch the graph of the function. Label local minimum, local maximum, and inflection points.



Example 3 Sketch a possible graph of a function f that satisfies the following conditions:

$$f''(x) < 0 \text{ on } (-\infty, -3), f''(x) > 0 \text{ on } (3, \infty).$$

$$f'(x) > 0 \text{ on } (-\infty, -5) \text{ or } (-1, \infty), f'(x) < 0 \text{ on } (-5, -1).$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty,$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

SOLUTION The best way to do this problem is to make a chart of increasing/decreasing and a concavity chart from the information above. For now, I am only going to put the graph of a possible function. There is no one answer to this problem.

