Practice

Summations

FORMULAS

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Find the sum.

① $$\sum_{k=0}^{3} 2^k = 2^0 + 2^1 + 2^2 + 2^3$$
$$= 1 + 2 + 4 + 8$$
$$= 15$$

② $$\sum_{k=0}^{3} k! = 0! + 1! + 2! + 3!$$
$$= 1 + 1 + 2\cdot 1 + 3\cdot 2\cdot 1$$
$$= 1 + 1 + 2 + 6$$
$$= 10$$

Use the formulas above to find the summation.

③ $$\sum_{i=1}^{100} (2i^2 - 3) = 2\sum_{i=1}^{100} i^2 - 3\sum_{i=1}^{100} 1$$
$$= 2\left(\frac{100(100+1)(2\cdot 100+1)}{6}\right) - 3(100$$
$$= \frac{100(101)(201)}{3} - 300$$
$$= 676{,}400$$

(4) $\sum_{i=1}^{100} (5i^3 - 3i + 1)$

$$= 5\sum_{i=1}^{100} i^3 - 3\sum_{i=1}^{100} i + \sum_{i=1}^{100} 1$$

$$= 5\left[\frac{(100)(100+1)}{2}\right]^2 - 3\,\frac{(100)(100+1)}{2} + 1\cdot 100$$

$$= 127{,}497{,}450$$

§ 11.7 Induction.

Prove by Induction.

(5) $S_n: 1^2 + 2^2 + 3^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Pf

$n = 1$ $\quad 1^2 = \dfrac{1(1+1)(2 \cdot 1 + 1)}{6}$

$1 = \dfrac{6}{6}$

$1 = 1$ ✓

Assume true for $n = k$. Show true for $n = k+1$.

$S_k: \quad 1^2 + 2^2 + 3^2 + \dots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

$+(k+1)^2 \qquad +(k+1)^2$

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}$$

$$= \frac{(k+1)\left[2k^2 + k + 6k + 6\right]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$S_{k+1}: \quad 1^2 + 2^2 + \dots + (k+1)^2 = \dfrac{(k+1)\left((k+1)+1\right)\left(2(k+1)+1\right)}{6}$ ✓

$\therefore$ S_n is true

⑥ T_n: $1^3+2^3+3^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

T_1 $n=1$ $\quad 1^3 = \left(\frac{1(1+1)}{2}\right)^2$

$1=1$

Assume true for $n=k$. Show true for $n=k+1$.

T_k: $1^3+2^3+3^3+\ldots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$

$+(k+1)^3 \qquad + (k+1)^3$

$$1^3+2^3+\ldots+(k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

T_{k+1}: $1^3+2^3+\ldots+(k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$ ✓

$\therefore$ T_n is true.

⑦ $S_n: 3+7+11+\dots+(4n-1) = n(2n+1)$

Pf

Base Case: $n=1$

$S_1:$

$$3 \stackrel{?}{=} 1(2\cdot 1+1)$$

$3=3$ yes ✓

Assume S_k is true $(n=k)$

show S_{k+1} is true $(n=k+1)$.

$S_k: \quad 3+7+11+\dots+(4k-1) = k(2k+1)$

$\qquad + 4(k+1)-1 \qquad + 4(k+1)-1$

$$3+7+11+\dots+4(k-1)+(4(k+1)-1)$$
$$= k(2k+1)+4(k+1)-1$$

$$= 2k^2+k+4k+4-1$$
$$= 2k^2+5k+3$$
$$= (2k+3)(k+1)$$

$S_{k+1} \quad 3+7+\dots+[4(k+1)-1] = (k+1)[2(k+1)+1]$ ✓

$\therefore$ S_n is true.

§11.5 Practice
The Binomial Theorem.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Evaluate.

(8) $\binom{5}{2} = \frac{5!}{(5-2)!(2!)} = \frac{5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(2\cdot1)}$

$= \frac{5\cdot4}{2} = 10$

(9) $\binom{7}{5} = \frac{7!}{(7-5)!5!} = \frac{7\cdot6\cdot\cancel{5\cdot4\cdot3\cdot2\cdot1}}{(2\cdot1)(\cancel{5\cdot4\cdot3\cdot2\cdot1})}$

$= \frac{7\cdot6}{2} = 7\cdot3 = 21$

(10) $\binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!}$

$= \frac{8\cdot7\cdot(6\cdot5\cdot4\cdot3\cdot2\cdot1)}{2\cdot1\,(6\cdot5\cdot4\cdot3\cdot2\cdot1)}$

$= \frac{8\cdot7}{2} = 4\cdot7 = 28$

(11) $\binom{100}{2} = \frac{100!}{2!(100-2)!} = \frac{100\cdot99\cdot98\cdots3\cdot2\cdot1}{(2\cdot1)(98\cdot97\cdots3\cdot2\cdot1)}$

$= \frac{(100)(99)}{2} = 50\cdot99$

$= 4950$

(12) $(x-2y)^4 =$

$$= x^4 + 4x^3(-2y) + 6x^2(-2y)^2 + 4x(-2y)^3 + (-2y)$$

$$= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(13) $(2a-3b)^5$

$$= (2a)^5 + 5(2a)^4(-3b) + 10(2a)^3(-3b)^2 + 10(2a)^2(-3b)^3 + 5(2a)(-3b)^4 + (-3b)$$

$$= 32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5.$$

(14) Write the first three terms of the binomial expansion.

$$(x-2y)^9 = \binom{9}{0}x^9 + \binom{9}{1}x^8(-2y) + \binom{9}{2}x^7(-2y)$$

$$= x^9 + 9x^8(-2y) + 36x^7 4y^2$$

$$= \boxed{x^9 - 18x^8 y + 144x^7y^2}$$

Aside

$$\binom{9}{2} = \frac{9!}{2\cdot 7!} = \frac{9\cdot 8}{2} = 36$$

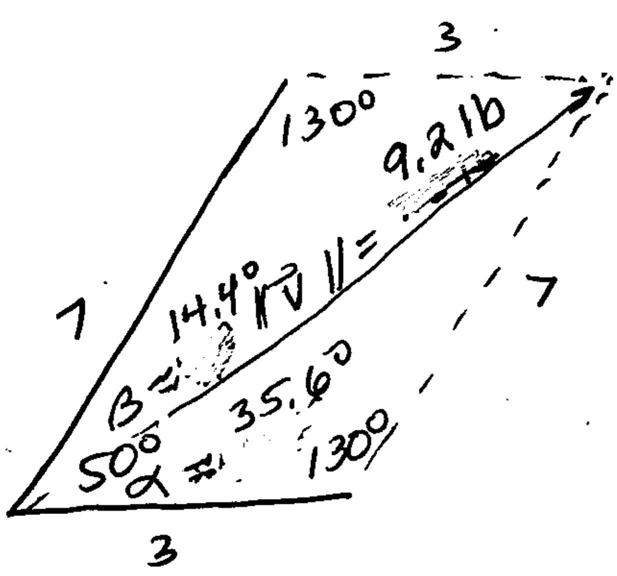
Vectors

(15) Find the magnitude of the resultant of the two force vectors. Then find the angle between the resultant and each force vector. The first force vector has magnitude 3 lb, and the second force vector has magnitude 7 lb. The angle between the two vectors is 50°.

SOLUTION

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$



- $\|\vec{v}\|^2 = 7^2 + 3^2 - 2(3)(7)\cos 130°$

$$\|\vec{v}\| = \sqrt{49 + 9 - 42\cos 130°}$$

$$\approx 9.2$$

- Find α

$$\frac{\sin\alpha}{7} = \frac{\sin 130°}{9.2}$$

$$\sin\alpha = \frac{7\sin 130°}{9.2}$$

$$\alpha = \sin^{-1}\left(\frac{7\sin 130°}{9.2}\right) \approx 35.6°$$

- $\beta = 50° - 35.6° \approx 14.4°$

(16) Sketch the conic. Label the intercepts. For hyperbolas, give the equation of the asymptotes and sketch the asymptotes.

$$25y^2 - 4x^2 = 100$$

Intercepts

$x = 0$

$25y^2 = 100$

$y^2 = 4$

$y = \pm 2$

$(0, \pm 2)$

$y = 0$

$-4x^2 = 100$

$x^2 = -25$

$x = \pm 5i$