Recall: $a+b$ is called a binomial

$(a+b)^n$

$n=2$ $(a+b)^2 = a^2+2ab+b^2$

$n=3$ $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b$

Pascal's Triangle

$$\begin{array}{ccccccccccc l}
 & & & & & 1 & & & & & & \\
 & & & & 1 & & 1 & & & & & \\
 & & & 1 & & 2 & & 1 & & & & n=2 \\
 & & 1 & & 3 & & 3 & & 1 & & & n=3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & n=4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & n=5
\end{array}$$

We use Pascal's Triangle to find the coefficients of $(a+b)^n$

The Binomial Theorem: If n is any positive integer, then for any real numbers a and b,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \quad \text{where} \quad \binom{n}{r} = \frac{n!}{r!(n-r)}$$

$$= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + \binom{n}{n} b^n$$

Proof:

$$(a+b)^n = \overbrace{(a+b)\cdot(a+b)\cdot(a+b)\cdot \cdots \cdot (a+b)}^{n \text{ times}}$$

When we expand using the distributive property, ~~we choose either~~ we will ~~have~~ choose either a or b from each factor.

How many ways can we get $a^{n-r}b^r$?

EXAMPLE $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$

How many ways can we get a^3b^2

$a \cdot a \cdot a \cdot b \cdot b$
$a \cdot a \cdot b \cdot a \cdot b$
$a \cdot b \cdot a \cdot a \cdot b$
$baaab$
etc

We are choosing 2 positions for b out of the five positions.
So there are $\binom{5}{2}$ ways to do this.

Note: $\binom{5}{2} = \frac{5!}{3!\,2!} = \binom{5}{3} = \frac{5\cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{3 \cdot 2 \cdot 1})(2 \cdot 1)} =$

EXAMPLE Write a complete binomial expansion.

(i) $(a-b)^4$

$= (a+(-b))^4$

$= a^4 + 4a^3(-b)^1 + 6a^2(-b)^2 + 4a(-b)^3 + 1(-b)^4$

$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

(ii) $(x-2y)^5 = (x+(-2y))^5$

$= x^5 - 5x^4(2y)^1 + 10x^3(2y)^2 - 10x^2(2y)^3 + 5x(2y)^4 - (2y)^5$

alternates pos, neg, pos

$= x^5 - 5\cdot 2\, x^4y + 10\cdot 4\, x^3y^2 - 10\cdot 8\, x^2y^3 + 5\cdot 16\, xy^5 - 32y^5$

$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^5 - 32y^5$

EXAMPLE Write the first three terms of the binomial expansion.

$$(ab^2 - 5c)^8 = (ab^2 + (-5c))^8 \qquad \underline{\underline{n=8}}$$

$$= \binom{8}{0}(ab^2)^8 + \binom{8}{1}(ab^2)^7(-5c) + \binom{8}{2}(ab^2)^6(-5c)^2$$

$\binom{8}{0} = 1$

$$\boxed{\text{Note: } \binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1}$$

$$= a^8 b^{16} + 8a^7 b^{14}(-5c) + \frac{8!}{2!(8-2)!} a^6 b^{12} (-5)^2 c$$

$$= a^8 b^{16} - 40 a^7 b^{14} c + 28 a^6 b^{12} \cdot 25 c^2$$

$$= \boxed{a^8 b^{16} - 40 a^7 b^{14} c + 700 a^6 b^{12} c^2}$$

Note: $\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)(n-2)\cdots 3 \cdot 2 \cdot}}{\cancel{(n-1)(n-2)}\cdots 3 \cdot 2} = n$

$$\binom{n}{1} = n$$

Aside $\binom{8}{2} = \frac{8!}{2!6!} = \frac{8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(2 \cdot 1)(\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})} = \frac{8 \cdot 7}{2} = 4 \cdot 7 = 28$