§ 11.7 Mathematical Induction

HW §11.7 # 1-35

Mathematical Induction is a proof technique. It is used to prove ascertions or statements that depend on a positive integer.

A <u>statement</u> is a sentence or equation that is either true or false.

<u>EXAMPLE</u>: The following is a statement.

$$S_n: \quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Principle of Mathematical Induction.

Let S_n be a statement for every positive integer n. If

1. S_1 is true, and (true for $n=1$)
2. the truth of S_K ($n=k$) implies the truth of S_{K+1} ($n=k+1$) for every positive integer K

then S_n is true for all positive integers n.

EXAMPLE ~~Proof~~. Prove by induction

$$S_n: \quad 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Pf

- Base case $(n=1)$

  Prove S_1: $1 = \frac{1(1+1)}{2}$

  $$1 = \frac{2}{1}$$

  $$1 = 1 \checkmark \text{ yes}$$

- Assume S_k is true $(n=k)$. Prove S_{k+1} $(n=k+1)$

$$S_k: \quad 1+2+3+\dots+k = \frac{k(k+1)}{2}$$

$$+(k+1) \qquad\qquad +(k+1)$$

$$1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$1+2+\dots\, k+(k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$1+2+\dots+k+(k+1) = \frac{(k^2+k)+(2k+2)}{2}$$

$$1+2+\dots+k+(k+1) = \frac{k^2+3k+2}{2}$$

$$1+2+\dots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

$n=k+1$ $$S_{k+1}: \quad 1+2+3+\dots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2} \checkmark \quad \square$$

Therefore S_n is true.

EXAMPLE Prove by Induction.

$$S_n: \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Pf

S_1 : $(n=1)$

$$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$$

$$1 = \frac{1\cdot 2\cdot 3}{6}$$

$$1 = \frac{6}{6}$$

$$1 = 1 \checkmark \text{ true.}$$

Show S_k implies S_{k+1}.

Assume S_k.

S_k : $(n=k)$

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$+(k+1)^2 \qquad\qquad +(k+1)^2$$

$$1^2+2^2+3^2+\dots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$= \frac{(k+1)\left(2k^2+k+6k+6\right)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$1^2+2^2+\dots\dots(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$S_{k+1}: \quad 1^2+2^2+3^2+\dots+k^2+(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \checkmark$$

Therefore S_n is true

§ 11.5 The Binomial Theorem

HW # 1-13, 45-67 odd

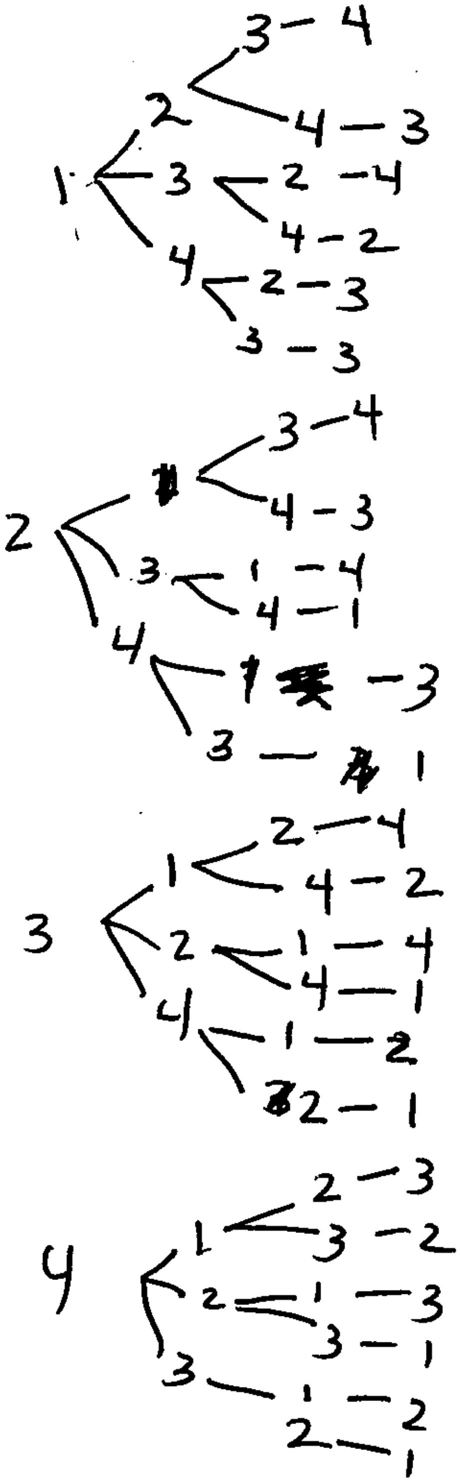
P_n is the number of permutations of n numbers. A permutation of a set of n elements S is a one-to-one mapping of the set S onto itself.

How many ways can we order the numbers $\{1, 2, 3\}$. That is, how many permutations are there of the set of numbers $\{1, 2, 3\}$.

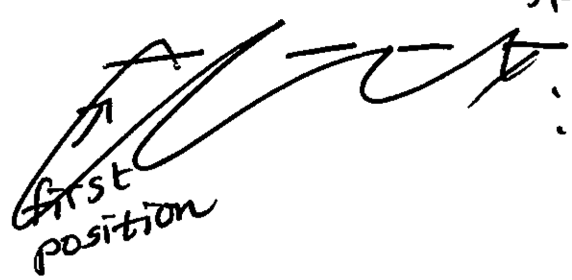
1, 2, 3	1 < 2 − 3
1, 3, 2	3 − 2
2, 1, 3	2 < 1 − 3
2, 3, 1	3 − 1
3, 1, 2	3 < 1 − 2
3, 2, 1	2 − 1

Answer: There are 6 permutations of $\{1, 2, 3\}$

EXAMPLE How many permutations are there of $\{1, 2, 3, 4\}$?



1, 2, 3, 4
1, 2, 4, 3
1, 3, 2, 4
1, 3, 4, 2
1, 4, 2, 3
1, 4, 3, 2
2, 1, 3, 4
2, 1, 4, 3
2, 3, 1, 4
2, 3, 4 1
2, 4, 1, 3
2, 4, 3, 1
⋮
4, 3, 2, 1



_ , _ , _ , _

first position 4 choices, 3 choices, 2 choices, 1 choice

Number of perm: $4 \cdot 3 \cdot 2 \cdot 1$

We write $|P_4| = 4\cdot3\cdot2\cdot1$
$= 24$

the bars mean the number of elements of the set P_4 (permutations of 4 numbers) or order of P_4.

<u>Definition</u> n factorial is

$$n! = n(n-1)(n-2)\cdots 3\cdot2\cdot1$$

<u>EXAMPLE</u> Evaluate:

a) $3! = 3\cdot2\cdot1 = 6$

b) $5! = 5\cdot4\cdot3\cdot2\cdot1 = (20)(6) = 120$

Theorem: The number of permutations of a set of n elements is $n!$. That is $|P_n| = n!$

$C(n,r)$ is the number of ways r elements can be chosen from a set of n elements.

<u>Theorem</u>: $C(n,r) = \dfrac{n!}{(n-r)!\,r!}$

for $0 \le r \le n$.

<u>Proof</u>:

$C(n,r)$ is the number of ways to choose r elements from a set of n elements.

~~of~~ $r!$ is the number of ~~ways~~ permutations of the r chosen elements.

$(n-r)!$ is the number ~~of~~ of permutations of the remaining $n-r$ elements.

$$C(n,r) \cdot r! \cdot (n-r)! = n!$$

$|P_n|$, the number of permutations of n elements

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

We denote $C(n, r)$ as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

called "n choose r".

EXAMPLE: Evaluate:

(a) $$\binom{5}{2} = \frac{5!}{2!\underbrace{(5-2)!}_{3!}} = \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{(2 \cdot 1)(\cancel{3 \cdot 2 \cdot 1})} = \frac{20}{2} = 10$$

b) $$\binom{20}{2} = \frac{20!}{2!(20-2)!} = \frac{20!}{2!\,18!} = \frac{20 \cdot 19 \cdot \cancel{18 \cdot 17 \cdots 3 \cdot 2 \cdot}}{(2 \cdot 1)(\cancel{18 \cdot 17 \cdot 16 \cdots})}$$

$$= \frac{(20)(19)}{2} = (10)(19) = 190$$

Formulas:

① $\cancel{\binom{n}{k} = \binom{n}{k}}$ $\binom{n}{k} = \binom{n}{n-k}$

Pf $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

EXAMPLE $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!}$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!}$$

② $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

~~Pf~~ $\cancel{\binom{n}{k-1} + \binom{n}{k}}$ Proof: excercise for the very enthusiastic.

This second formula implies that Pascal Triangle gives the ~~the~~ coefficients for $(a+b)^n$;

because $\binom{0}{0} = \frac{0!}{0!\,0!} = \frac{1}{1\cdot 1}$ is the first line

Note: by convention $0! = 1$

The second line is $\binom{1}{0}$ $\binom{1}{1}$

which gives $\binom{1}{0} = \frac{1!}{0!\,1!} = 1$

$\binom{1}{1} = \frac{1!}{0!\,1!} = 1$

We get

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\underset{K=0}{\binom{2}{0}} \quad \underset{K=1}{\binom{2}{1}} = 2 \quad \underset{K=2}{\binom{2}{2}} = 1 \qquad n=2$$

$$= 1 +$$

$$\binom{3}{0} = 1 \quad \binom{3}{1}\ K=1 = 3 \quad \binom{3}{2} = 3 \quad \binom{4}{2} = 1 \qquad n=3$$

$$\underbrace{\binom{2}{0}}_{K=0} + \underbrace{\binom{2}{1}}_{K=1} = \binom{3}{1}$$

$n=2$ $\quad$ $n+1=3$

formula ~~proper~~ $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$