Theorem:

If $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

and $\sum_{i=1}^{n} b_i = b_1 + b_2 + b_3 + \cdots + b_n$

are summations

and c is a constant, then

① $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

Proof: $\sum_{i=1}^{n} c a_i = c a_1 + c a_2 + \cdots + c a_n$
$= c(a_1 + a_2 + \cdots + a_n)$
$= c \sum_{i=1}^{n} a_i$

② $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$

Pf

$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = (a_1 + a_2 + a_3 + \cdots + a_n)$
$+ (b_1 + b_2 + b_3 + \cdots + b_n)$
$= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$
$= \sum_{i=1}^{n} (a_i + b_i)$

FORMULAS

(1) $$\sum_{i=1}^{n} 1 = n$$

Pf $$\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\dots+1}_{n \text{ times}} = n$$

$$\sum_{i=1}^{n} c = cn$$

Pf $$\sum_{i=1}^{n} c = \underbrace{c+c+c+\dots+c}_{n \text{ times}} = nc$$

(2) $$\sum_{i=1}^{n} i = 1+2+3+\dots+n$$
$$= \frac{n(n+1)}{2}$$

Example Find the sum.

$1+2+3+\dots+100$

SOLUTION: $n = 100$

$$= \frac{100(100+1)}{2} = 50(101) = 5{,}050$$

Pf (idea of proof

$$1 + 2 + 3 + 4 + 50 + 51 + \dots + 97 + 98 + 99 + 100$$

101

100

101

101

101

$$= 101(50) = 101\left(\frac{100}{2}\right) = \frac{(n+1)n}{2}$$

$$= \frac{n(n+1)}{2}$$

③ $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

④ $$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3$$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

EXAMPLE Find the sum.

(1) $$\sum_{i=1}^{100}\left(i^2+2i+5\right)$$

SOLUTION: $$\sum_{i=1}^{100} i^2 + \sum_{i=1}^{100} 2i + \sum_{i=1}^{100} 5$$

$$= \sum_{i=1}^{100} i^2 + 2\sum_{i=1}^{100} i + \sum_{i=1}^{100} 5$$

$n = 100$

Use formulas.

$$= \frac{(100)(100+1)(2\cdot 100+1)}{6} + 2\left(\frac{(100)(100+1)}{2}\right)$$

$$+ 5\cdot 100$$

$$= \frac{(100)(101)(201)}{6} + \cancel{2}\,(100)(101) + 500$$

$$= 348{,}950$$