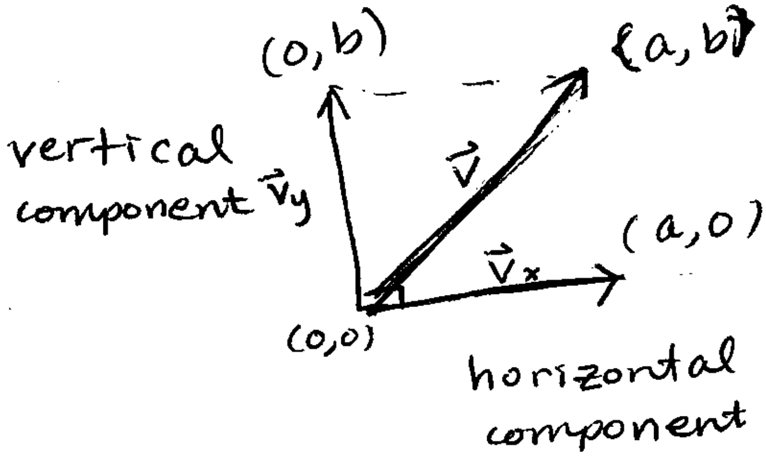
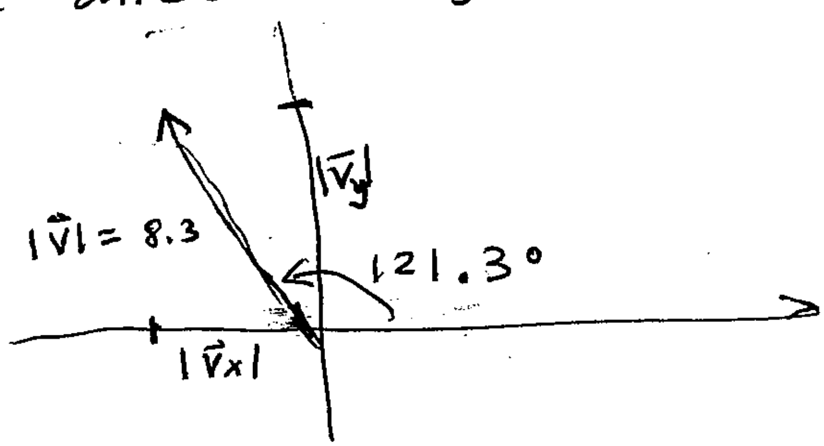
§7.3 Continued

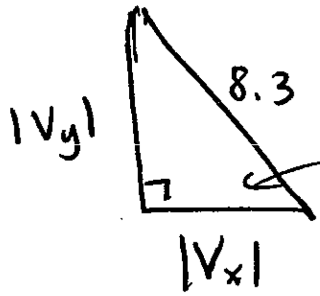
Horizontal and Vertical Components



If $\vec{v} = \langle a, b \rangle$ then $\vec{v}_x = \langle a, 0 \rangle$
and $\vec{v}_y = \langle 0, b \rangle$

EXAMPLE Find the magnitude (length) of the horizontal and vertical component for a vector $\vec{v}$ with magnitude 8.3 and direction angle 121.3°.





$180 - 121.3 = 58.7°$

$$\sin 58.7° = \frac{opp}{hyp} = \frac{|V_y|}{8.3}$$

$$|V_y| = 8.3 \sin 58.7° = 7.1$$

$$\cos 58.7° = \frac{adj}{hyp} = \frac{|V_x|}{8.3}$$

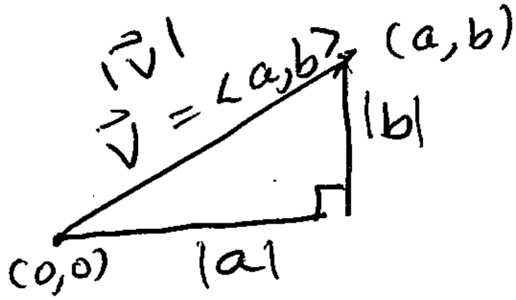
$$|V_x| = 8.3 \cos 58.7°$$

$$\approx 4.3$$

The Direction Angle is the angle $\vec{v}$ makes with the positive angle. It is between 0° and 360°.

The Magnitude (length, norm) of a vector $\vec{v} = \langle a, b \rangle$ is ~~given~~ given by

$$|\vec{v}| = \sqrt{a^2+b^2}$$



$$|\vec{v}|^2 = |a|^2 + |b|^2$$

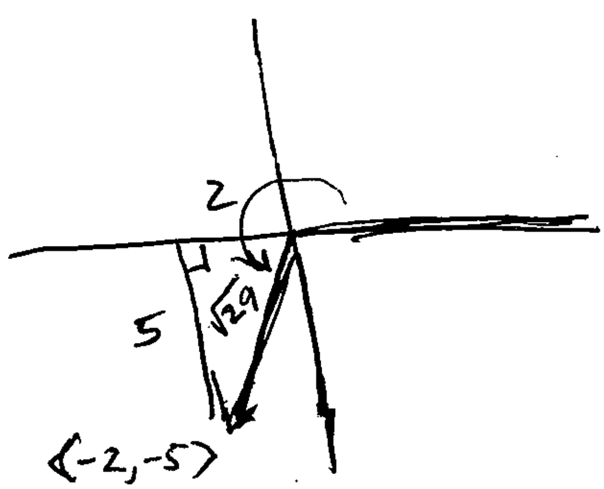
$$|\vec{v}|^2 = a^2 + b^2$$

$$|\vec{v}| = \sqrt{a^2+b^2}$$

EXAMPLE Find the magnitude and direction angle of the vector $\vec{v} = \langle -2, -5 \rangle$.

SOLUTION: $|\vec{v}| = \sqrt{(-2)^2 + (-5)^2}$

$= \sqrt{4 + 25} = \sqrt{29}$



$\tan \theta_R = \frac{\text{opp}}{\text{adj}} = \frac{5}{2}$

$\theta_R = \tan^{-1}\left(\frac{5}{2}\right) \approx 68.2°$

Direction $180° + 68.2°$

$= 248.2°$

The Dot Product

$\vec{v} = \langle a, b \rangle, \ \vec{w} = \langle c, d \rangle$

$$\vec{v} \cdot \vec{w} = ac + bd$$

dot

EXAMPLE Evaluate:

(i) $\langle 2, 3 \rangle \cdot \langle -2, 5 \rangle$

$= (2)(-2) + (3)(5)$

$= -4 + 15$

$= 11$

(ii) $\langle -7, 5 \rangle \cdot \langle 2, -3 \rangle = (-7)(2) + (5)(-3)$

$= -14 - 15 = -29$

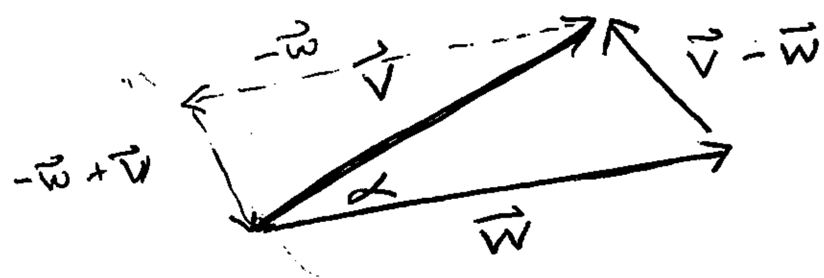
We have

$\vec{v} \cdot \vec{v} = \langle a, b \rangle \cdot \langle a, b \rangle = a^2 + b^2 = |\vec{v}|^2$

$$\boxed{\vec{v} \cdot \vec{v} = |\vec{v}|^2}$$

__Theorem__ If $\vec{v}$ and $\vec{w}$ are nonzero vectors and α is the angle between them, then

$$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$



__Proof__:

Use Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$|\vec{v}-\vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\alpha$$

$$(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2|\vec{v}||\vec{w}|\cos\alpha$$

if $\vec{v} = \langle v_1, v_2\rangle$, $\vec{w} = \langle w_1, w_2\rangle$ we get

$$\langle v_1-w_1, v_2-w_2\rangle\cdot\langle v_1-w_1, v_2-w_2\rangle$$
$$= \langle v_1, v_2\rangle\cdot\langle v_1, v_2\rangle + \langle w_1, w_2\rangle\cdot\langle w_1,$$
$$-2|\vec{v}||\vec{w}|\cos\alpha$$

$$(v_1-w_1)^2 + (v_2-w_2)^2 = (v_1^2+v_2^2) + (w_1^2+w_2)^2 - 2|\vec{v}||\vec{w}|\cos\alpha$$

$$\cancel{v_1^2} - 2v_1w_1 + \cancel{w_1^2} + \cancel{v_2^2} - 2v_2w_2 + \cancel{w_2^2} = \cancel{v_1^2} + \cancel{v_2^2} + \cancel{w_1^2} + \cancel{w_2^2} - 2|\vec{v}||\vec{w}|\cos\alpha$$

$$-2v_1w_1 - 2v_2w_2 = -2|\vec{v}||\vec{w}|\cos\alpha$$

$$\cancel{-2}(v_1w_1 + v_2w_2) = \cancel{-2}|\vec{v}||\vec{w}|\cos\alpha$$

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\alpha$$

$$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

The Cauchy-Schwartz Inequality

$$|\vec{v}\cdot\vec{w}| \leq |\vec{v}||\vec{w}|$$

<u>Pf</u>

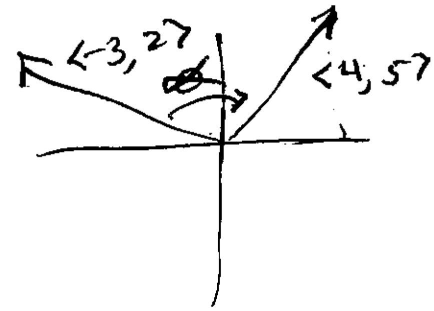
$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\alpha$$

$$|\vec{v}\cdot\vec{w}| = \big||\vec{v}||\vec{w}|\cos\alpha\big|$$

$$= |\vec{v}||\vec{w}|\,|\cos\alpha|$$

$$\leq |\vec{v}||\vec{w}|\cdot 1 = |\vec{v}||\vec{w}|$$

EXAMPLE Find the smallest positive angle between each pair of vectors.



(a) $\langle -3, 2\rangle, \langle 4, 5\rangle$

$$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

Find α

$$\cos\alpha = \frac{\langle -3, 2\rangle \cdot \langle 4, 5\rangle}{|\langle -3, 2\rangle|\;|\langle 4, 5\rangle|}$$

$$= \frac{(-3)(4) + (2)(5)}{\left(\sqrt{(-3)^2+(2)^2}\right)\left(\sqrt{4^2+5^2}\right)}$$

$$\cos\alpha = \frac{-2}{\sqrt{13}\,\sqrt{41}}$$

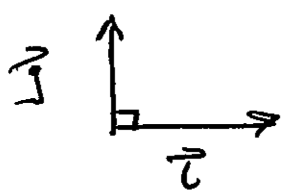
16
25
41

$$\alpha = \cos^{-1}\left(\frac{-2}{\sqrt{13}\,\sqrt{41}}\right) \approx 94.97^\circ$$

Unit Vectors

$\vec{i} = \langle 1, 0 \rangle, \quad \vec{j} = \langle 0, 1 \rangle$

Note $|\vec{i}| = 1, \quad |\vec{j}| = 1.$



Note for any vector $\vec{v} = \langle a, b \rangle$
we have

$$\begin{aligned} \vec{v} = \langle a, b \rangle &= \langle a, 0 \rangle + \langle 0, b \rangle \\ &= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle \\ &= a\vec{i} + b\vec{j} \end{aligned}$$

EXAMPLE Write each vector as a linear combination of the unit vectors $\vec{i}, \vec{j}$.

(a) $\langle -2, 6 \rangle$

SOL'N $= -2\vec{i} + 6\vec{j}$

(b) $\langle -4, -1 \rangle = -4\vec{i} - \vec{j}$

Thm Two vectors $\vec{v}$ and $\vec{w}$ are perpindicular (orthogonal) if and only if $\vec{v} \cdot \vec{w} = 0$

Pf $\vec{v}$ and $\vec{w}$ orthogonal

iff the angle between them is $\alpha = 90^\circ$

iff $\cos\alpha = 0$

We have

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||w|} = \cos\alpha$$

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||w|} = 0$$

iff $\vec{v} \cdot \vec{w} = 0$

EXAMPLE
Determine whether $\vec{v} = \langle 2, -3 \rangle$ and $\vec{w} = \langle 3, 2 \rangle$ are orthogonal.

Sol'n $\langle 3, 2 \rangle \cdot \langle 2, -3 \rangle$
$= (3)(2) + (2)(-3) = 0$
Yes, orthogonal.

Two vectors are parallel

iff $\cos \alpha = \pm 1$

$\cos 180° = -1$

$\alpha = 0°$

$\cos 0° = 1.$