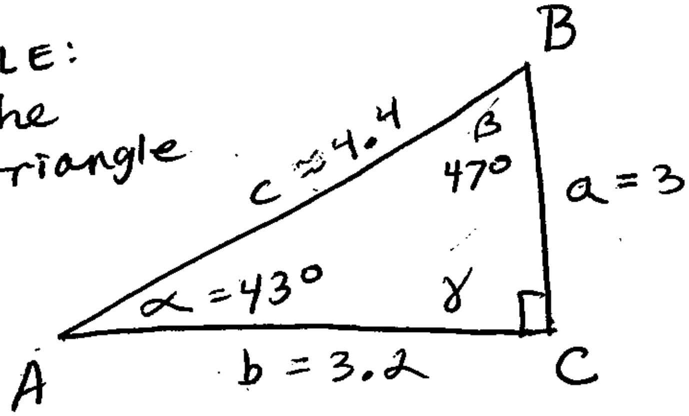
§5.6 Notes (Continued)

Solving Right Triangles.

EXAMPLE:
Solve the right triangle



Given
$\beta = 47°$, $a=3$

Solution:

- The sum of the angles of a triangle is 180°.
- The sum of the acute angles in a right triangle is 90°.

Solve for α

$$\alpha + \beta = 90°$$
$$\alpha = 90° - \beta$$
$$\alpha = 90° - 47° = 43°$$

Solve for c

$$\sin\alpha = \frac{opp}{hyp} = \frac{a}{c}$$

$$\sin 43° = \frac{3}{c}$$

$$c \sin 43° = 3$$

$$c = \frac{3}{\sin 43°} \approx 4.4$$

Solve for b

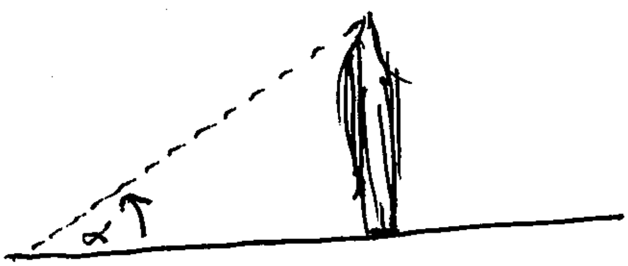
$$\tan\alpha = \frac{opp}{adj} = \frac{a}{b}$$

$$\tan 43° = \frac{3}{b}$$

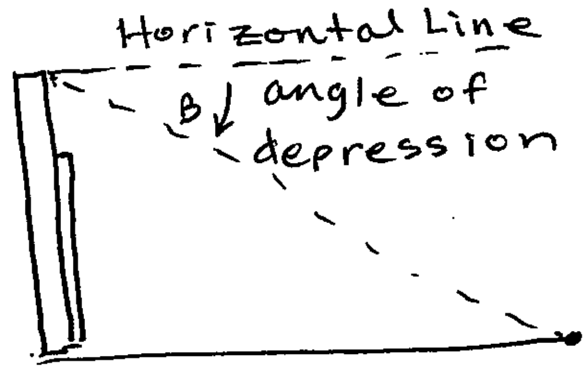
$$b\tan 43° = 3$$

$$b = \frac{3}{\tan 43°} \approx 3.2$$

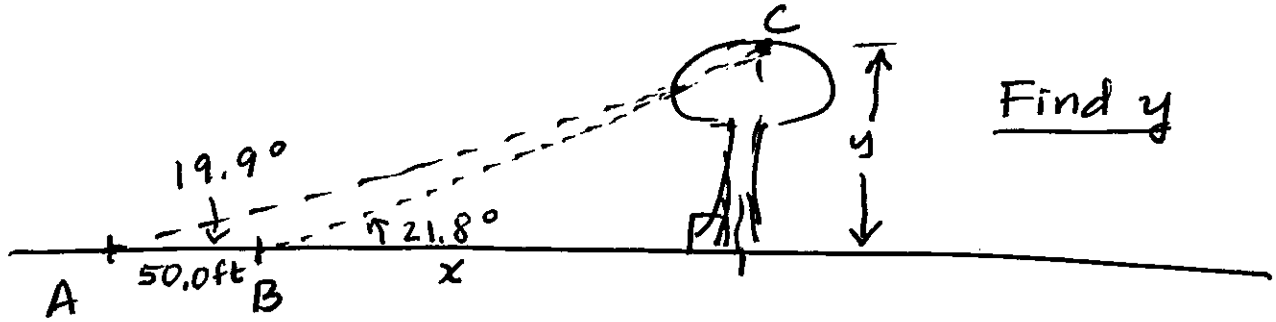
Applications:



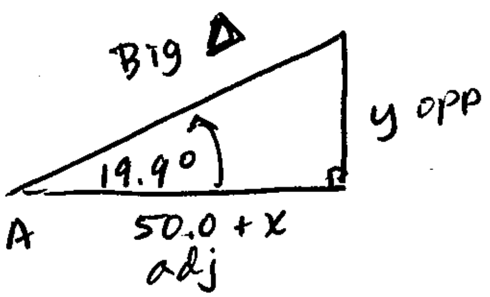
Angle of elevation

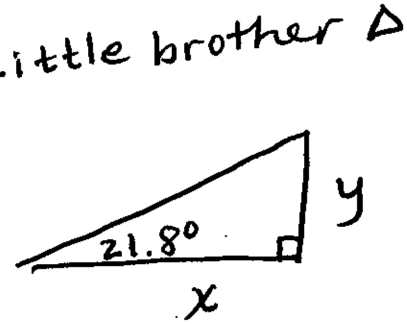


EXAMPLE The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?



Find y





$$\tan 21.8° = \frac{y}{x}$$

$$\tan\theta = \frac{opp}{adj}$$

$$\tan 19.9° = \frac{y}{50.0+x}$$

Two equations, two unknown.

$(50.0+x)\tan 19.9° = y$ $\qquad$ $x\tan 21.8° = y$

Solve for x:

$$(50.0+x)\tan 19.9° = x\tan 21.8°$$

$$50.0\tan 19.9° + x\tan 19.9° = x\tan 21.8°$$

$$x\tan 19.9° - x\tan 21.8° = -50.0\tan 19.$$

$$x(\tan 19.9° - \tan 21.8°) = -50.0\tan 19.9°$$

$$x = \frac{-50.0 \tan 19.9^\circ}{\tan 19.9^\circ - \tan 21.8^\circ}$$

Solve for y

$$y = x \tan 21.8^\circ$$

$$= \left(\frac{-50.0 \tan 19.9^\circ}{\tan 19.9^\circ - \tan 21.8^\circ} \right) \tan 21.8^\circ$$

$$y \approx 191 \text{ ft}$$

§6.3 Notes (Continued)

FORMULA
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Let's Find a formula for $\cos(\alpha-\beta)$.

$$\cos(\alpha-\beta) = \cos(\alpha+(-\beta)) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

Recall: $\cos(-\beta) = \cos\beta$
$\sin(-\beta) = -\sin\beta$

$$= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta)$$

FORMULA
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Cofunction Identities

Let's put $\alpha = \pi/2$ into the identity for $\cos(\alpha-\beta)$

$$\cos\left(\frac{\pi}{2}-\beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$
$$= 0\cdot\cos\beta + 1\cdot\sin\beta$$

$$\boxed{\cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta}$$

If we now let $\beta = \frac{\pi}{2} - \alpha$ we get

$$\cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\alpha\right)\right) = \sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\boxed{\cos\alpha = \sin(\pi/2 - \alpha)}$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\sec\left(\frac{\pi}{2} - u\right) = \csc u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Sine of a Sum or Difference

We now derive a formula for $\sin(\alpha + \beta)$.

$$\sin(\alpha + \beta) = \overbrace{\cos\left(\frac{\pi}{2} - (\alpha + \beta)\right)}^{\text{cofunction Identity}}$$

$$= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin(\beta)$$

$$= \overbrace{\sin\alpha\cos\beta + \cos\alpha\sin\beta}^{\text{cofunction identity}}$$

Identity

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

Let's now derive an identity for $\sin(\alpha - \beta)$

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

use
$\cos(-\beta) = \cos\beta$
$\sin(-\beta) = -\sin\beta$

Identity

$$\boxed{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Test problem

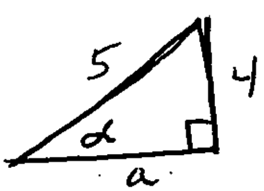
EXAMPLE Suppose $\sin\alpha = \frac{4}{5}$ and $\cos\beta = -\frac{12}{13}$, where α is in quadrant I and β is in quadrant II.

(i) Find the exact value of $\sin(\alpha+\beta)$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right) + \left(\underset{?}{\frac{3}{5}}\right)\left(\underset{?}{\frac{5}{13}}\right)$$

- if $\sin\alpha = \frac{4}{5} = \frac{opp}{hyp}$
  α in Q I,
  find $\cos\alpha$.



$a^2+4^2=5^2$

$a^2=5^2-4^2$

$=25-16=9$

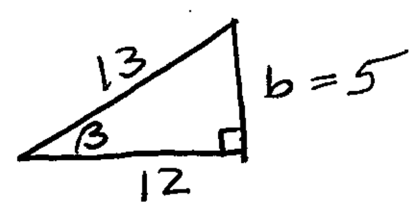
$a^2=9$

$a=3$

$\cos\alpha = \frac{adj}{hyp} = \frac{3}{5}$ ✓

pos because quad 1

- Find $\sin\beta$
  if $\cos\beta = \frac{-12}{13} = \frac{adj}{hyp}$
  β in QII.



$12^2+b^2=13^2$

$b^2=13^2-12^2$

$=169-144$

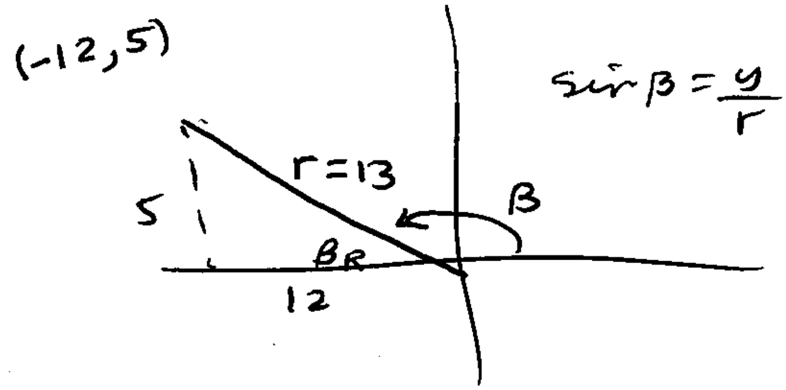
$b^2=25$

$b=5$

$\sin\beta = \frac{opp}{hyp}$

$= \frac{5}{13}$ ← pos in QII

OR



$\sin\beta = \frac{y}{r} = \frac{5}{13}$

$$\sin(\alpha+\beta) = \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$$

$$= \frac{-48}{65} + \frac{15}{65} = \frac{-33}{65}$$

$$\begin{array}{r} -48 \\ 15 \\ \hline 33 \end{array}$$

(ii) Find $\cos(\alpha+\beta)$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(\frac{3}{5}\right)\left(\frac{-12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)$$

$$= \frac{-36}{65} - \frac{20}{65}$$

these numbers are from part (i)

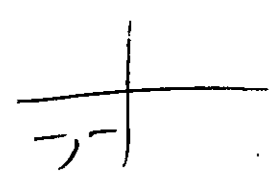
$$= \frac{-56}{65}$$

(iii) In which quadrant does $\alpha+\beta$ lie?

SOLUTION

$$\cos(\alpha+\beta) = \frac{-56}{65}$$

$$\sin(\alpha+\beta) = -\frac{33}{65}$$



$\alpha+\beta$ lies in quadrant III.

§6.4 Double-Angle and Half-Angle Identities. HW §6.4 #1–51 odd

① $\sin 2\theta = 2\sin\theta\cos\theta$

We use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Pf $\sin(2\theta) = \sin(\theta+\theta)$

$= \sin(\theta)\cos\theta + \cos\theta\sin\theta$

$= 2\sin\theta\cos\theta$

② a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

we use $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Pf $\cos(2\theta) = \cos(\theta+\theta)$

$= \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

b) $\cos 2\theta = 1 - 2\sin^2\theta$

Pf $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(use $\cos^2\theta + \sin^2\theta = 1$
$\cos^2\theta = 1 - \sin^2\theta$)

$= (1-\sin^2\theta) - \sin^2\theta$

$= 1 - 2\sin^2\theta.$

c) $\cos 2\theta = 2\cos^2\theta - 1$

Use $\sin^2\theta = 1 - \cos^2\theta$

Pf

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \cos^2\theta - (1-\cos^2\theta)$

$= \cos^2\theta - 1 + \cos^2\theta$

$= 2\cos^2\theta - 1$