§4.3 Rules of Logarithms

HW §4.3 # 1–83 odd

<u>LAWS OF LOGS</u> For $M>0$ and $N>0$ and any real number r, then

1. $\log_a(MN) = \log_a M + \log_a N$
2. $\log_a(M/N) = \log_a(M) - \log_a(N)$
3. $\log_a(M^r) = r\log_a(M)$

<u>Example</u>: Write in terms of $\ln 2$, $\ln 3$, and $\ln 5$.

a) $\ln(12) = \ln(2\cdot 2\cdot 3) = \ln(2^2\cdot 3)$
$$= \ln(2^2) + \ln(3)$$
$$= 2\ln 2 + \ln 3$$

b) $\ln(100) = \ln(4\cdot 25) = \ln(2^2\cdot 5^2)$
$$= \ln 2^2 + \ln 5^2$$
$$= 2\ln 2 + 2\ln 5$$

EXAMPLE Write as a single logarithm.

a) $\ln 8 - \ln 2$

$$= \ln(8/2) = \ln 4$$

b) $3\ln 2 + 2\ln 3$

$$= \ln(2^3) + \ln(3^2)$$

$$= \ln(2^3 \cdot 3^2) = \ln(8 \cdot 9)$$

$$= \ln 72$$

Example: Expand the logarithm.

a) $\ln(x^2 y^3)$

$$= \ln(x^2) + \ln(y^3)$$

$$= 2\ln x + 3\ln y$$

b) $\ln\left(\dfrac{x^2}{y^5 z^7}\right) = \ln(x^2) - \ln(y^5 z^7)$

$$= \ln(x^2) - (\ln y^5 + \ln z^7)$$

$$= \ln x^2 - \ln y^5 - \ln z^7$$

$$= 2\ln x - 5\ln y - 7\ln 7$$

c) $\ln\left(x\sqrt{x+1}\right) = \ln\left(x(x+1)^{1/2}\right)$

$= \ln x + \ln(x+1)^{1/2}$

$= \ln x + \frac{1}{2}\ln(x+1)$

Note $\ln_e(1) = 0$ because $e^0 = 1$

$\ln(2) = \ln(1+1)$

$\neq \ln(1) + \ln(1)$

$= 0 + 0 = 0$

so $\ln(A+B) \neq \ln A + \ln B$

Cancellation Properties

① $\log_a(a^x) = x$

② $a^{\log_a(x)} = x$

This is true because
$f(x) = a^x$ and $g(x) = \log_a x$
are inverse functions
so that $f(g(x)) = x$
and $g(f(x)) = x$

In particular
$$\ln(e^x) = x$$
and $$e^{\ln x} = x$$

Other Rules

- $\log_a a = 1$ because $a^1 = a$
- in particular $\ln e = 1$
- $\log_a(1) = 0$ because $a^0 = 1$
- in particular $\ln(1) = 0$.

Base - Change Formula:

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$$

EXAMPLE: Write in terms of natural log or common log.

① $\log_2 5$

Solution $\ln_2 5 = \frac{\ln 5}{\ln 2}$

② $\log_3 7 =$

SOLUTION $\frac{\ln 7}{\ln 3} = (\ln 7)/(\ln 3)$

③ $\log_3 5 = \frac{\log 5}{\log 3}$

Proof Let $x = \log_a(M)$

Write as an exponent

$$a^x = M$$

Apply $\log_b$ to both sides

$$\log_b a^{x} = \log_b M$$

Use property of log to bring x to the front.

$$x \log_b a = \log_b M$$

$$x = \frac{\log_b M}{\log_b a}$$

§4.4 More Equations and Applications

HW §4.4 # 1–41 odd, 61–65 odd

<u>EXAMPLE</u>: Solve ① $\log(x-3) = 4$

<u>SOLUTION</u> • write as an exponent.

$$\log_{10}(x-3) = 4$$
$$10^4 = x-3$$
$$10000 = x-3$$
$$x = 10003$$

② $\log_2 x + \log_2(x+2) = \log_2(6x+5)$

<u>SOLUTION</u> Combine the logs

$$\log_2 x(x+2) = \log_2(6x+5)$$
$$x(x+2) = (6x+5)$$
$$x^2+2x = 6x+5$$
$$x^2-4x-5 = 0$$
$$(x-5)(x+1) = 0$$
$$x = 5, \cancel{x=-1}$$

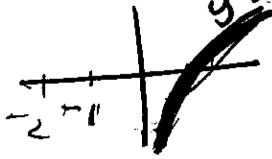
<u>check</u> $x=5$

$$\left.\begin{matrix} ln_2(5) \\ ln_2(5+2) \\ ln_2(6\cdot5+5) \end{matrix}\right\} \underline{ok}$$

<u>check</u> $x=-1$

$ln_2(-1)$ <u>undef</u>

$y = \log_2 x$ $\qquad 2^y = x$



Dom $x > 0$

③ $2 \ln x = \ln(x+3) + \ln(x-1)$

SOLUTION

Combine logarithms

$$2 \ln x = \ln(x+3)(x-1)$$

$$\ln x^2 = \ln(x+3)(x-1)$$

$$x^2 = (x+3)(x-1)$$

$$x^2 = x^2 + 2x - 3$$

$$0 = 2x - 3$$

$$3 = 2x, \quad x = 3/2$$

Note

$\ln(3/2)$ defined

$\ln(3/2+3)$ defined

$\ln(3/2-1) = \ln 1/2$ defined

Equations involving exponents.

EXAMPLE Solve

(1) $2^x = 5$

SOLUTION Apply ln to both sides.

$$\ln(2^{x}) = \ln(5)$$

$$x \ln 2 = \ln 5$$

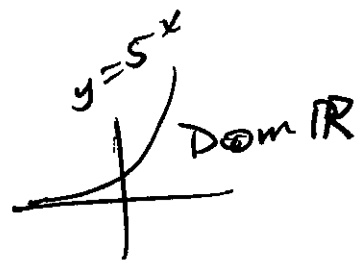
$$x = \frac{\ln 5}{\ln 2}$$

(2) $5^{3x} = 7$

SOLUTION $\ln(5^{3x}) = \ln 7$

$$3x \ln 5 = \ln 7$$

$$x = \frac{\ln 7}{3 \ln 5}$$



(3) $5^x = 7^{(x+3)}$

$$\ln 5^{x} = \ln 7^{(x+3)}$$

$$x \ln 5 = (x+3)\ln 7$$

$$x \ln 5 - x \ln 7 = 0$$

$$x \ln 5 = x \ln 7 + 3 \ln 7$$

$$x \ln 5 - x \ln 7 = 3 \ln 7$$

$$x(\ln 5 - \ln 7) = 3 \ln 7$$

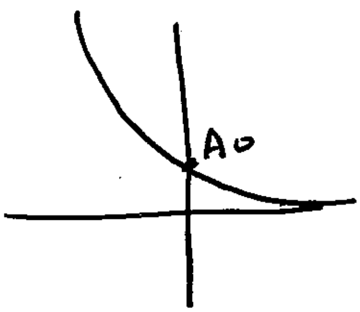
$$x = \frac{3 \ln 7}{\ln 5 - \ln 7}$$

Radioactive Dating

Radioactive decay is given by the formula

$$A = A_0 e^{-kt}$$

where A_0 is the begining amount
A is the final amount.
k is a positive constant.
t is time.



The half-life, h, is the time it takes for half of the radioactive substance to decay. This is a constant depending on the element.

Let's find a formula for k in terms of h.

Let $A = \frac{1}{2}A_0$. Let time $t=h$.

We get $\frac{1}{2}A_0 = A_0 e^{-kh}$

$$\frac{1}{2} = e^{-kh}$$

$$\ln(1/2) = \ln e^{-kh}$$

$$\ln(1/2) = -kh$$

$$\frac{\ln(1/2)}{-h} = k$$