§ 3.2 Zeros of Polynomial Functions

HW § 3.2 # 1 – 71 odd

Long Division

$358 \div 7$

$$\begin{array}{r} 51 \text{ r. } 1 \\ 7 \overline{)358} \\ -35\downarrow \\ \hline 08 \\ -7 \\ \hline 1 \end{array}$$

$\div, \times, -, \downarrow$

$358 \div 7 = 51 + \frac{1}{7}$

$\frac{358}{7} = 51 + \frac{1}{7}$

$358 = 51 \cdot 7 + 1$

remainder

Division Algorithm.

If m & n are integers, there exists integers q and r such that

$$n = mq + r$$

where $0 \le r < m$

Long Division of Polynomials.

$$(x^3 + 2x^2 - 3x + 5) \div (x+2)$$

$$
\begin{array}{r l}
 & \quad x^2 \qquad\quad -3 \quad r. 11 \\
x+2 & \overline{\big)\, x^3 + 2x^2 - 3x + 5} \\
 & \underline{-x^3 + {}^{-}2x^2} \ \downarrow \\
 & \qquad\quad 0 \quad -3x + 5 \\
 & \qquad\qquad \underline{+3x + 6} \\
 & \qquad\qquad\qquad 11
\end{array}
$$

$$\frac{x^3 + 2x^2 - 3x + 5}{x+2} = x^2 - 3 + \frac{11}{x+2}$$

$$x^3 + 2x^2 - 3x + 5 = (x^2 - 3)(x+2) + \underset{\text{remainder}}{\underset{\uparrow}{11}}$$

Division Algorithm.

If $N(x)$ and $M(x)$ are polynomials then there exist polynomials $Q(x)$ and $R(x)$ such that

$$N(x) = M(x)\,Q(x) + R(x)$$

where $\qquad 0 \le \deg R < \deg M$

Synthetic Division

$$(x^3+2x^2-3x+5) \div (x+2)$$

$$\begin{array}{r|rrrr}
-2 & 1 & 2 & -3 & 5 \\
 & & + & & \\
 & \downarrow & \xrightarrow{1*-2} -2 & \xrightarrow{0*-2} 0 & \xrightarrow{-3*-2} 6 \\
\hline
 & 1 & 0 & -3 & 11
\end{array}$$

$$1\cdot x^2 + 0\cdot x \; -3$$

$$1\cdot x^2 \; -3 \quad R.\,11$$

$$x^2-3 \quad R.\,11$$

The division algorithm can be written as

$$P(x) = Q(x)\,(x-c) + R$$

where we are dividing $P(x)$ by $(x-c)$. we have R is a number.

<u>Remainder Theorem</u>: If R is the remainder when $P(x)$ is divided by $x-c$, then

$$R = P(c).$$

<u>Pf</u>

$$\begin{aligned}
P(x) &= Q(x)(x-c)+R \\
P(c) &= Q(c)(c-c)+R \\
&= 0+R \\
P(c) &= R
\end{aligned}$$

Example: Let $P(x) = 3x^3 + x^2 + 5$. Use the Remainder Theorem to find $P(2)$.

SOLUTION

$$\begin{array}{r|rrrr} 2 & 3 & 1 & 0 & 5 \\ & \downarrow & \overset{3\times 2}{6} & \overset{7\cdot 2}{14} & \overset{14\cdot 2}{28} \\ \hline & 3 & 7 & 14 & 33 \end{array}$$

$R = 33$

$P(2) = 33$

check:
$$\begin{aligned} P(2) &= 3(2)^3 + (2)^2 + 5 \\ &= 3\cdot 8 + 4 + 5 \\ &= 24 + 4 + 5 = 33 \end{aligned}$$

Definition: The number c is a zero of a polynomial $P(x)$ if $P(c) = 0$.

The Factor Theorem The number c is a zero of the polynomial $y = P(x)$ if and only if $x - c$ is a factor of $P(x)$.

Proof: Assume $x - c$ is a factor of $P(x)$. Then $P(x) = (x-c)\,Q(x)$

so $P(c) = (c-c)\,Q(x) = 0$.

so c is a zero.

Assume c is a zero of $P(x)$.

By the div. alg.

$$P(x) = (x-c)\,Q(x) + R$$

for some $Q(x)$ and number R.

But $P(c)=0$ so

$$0 = P(c) = (c-c)\,Q(c) + R$$

$$0 = R$$

so $P(x) = (x-c)\,Q(x)$

EXAMPLE Determine whether $x+4$ is a factor of $P(x) = x^3 - 13x + 12$. If it is a factor, then factor $P(x)$ completely.

SOLUTION $x+4$ is a factor if $x=-4$ is a zero of $P(x)$.

$$P(-4) = (-4)^3 - 13(-4) + 12$$

$$= -64 + 52 + 12 = 0$$

so yes $(x+4)$ is a factor.

Let's factor completely. Divide $x^3-13x+12$ by $x+4$.

$$\begin{array}{r|rrrr} -4 & 1 & 0 & -13 & +12 \\ & \downarrow & -4 & 16 & -12 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

$R=0$

$$P(x) = (x+4)(x^2-4x+3) + 0$$

$$\boxed{P(x) = (x+4)(x-3)(x-1)}$$

The Fundamental Theorem of Algebra

If $y=P(x)$ is a polynomial of positive degree, then $y=P(x)$ has at least one zero in the set of complex numbers.

The Rational Zero Theorem.

If $f(x)=a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial, and p/q is a rational number in lowest terms that is a zero for $f(x)$, then p is a factor of a_0 and q is a factor of a_n.

EXAMPLE Find all the possible rational zeros for the polynomial.

a) $f(x) = \underset{\text{lead coeff}}{2}x^3 - 11x + \underset{\text{constant coefficient}}{6}$

Factors of 6: $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of 2: $\pm 1, \pm 2$

$\dfrac{\text{Factors of Const. Coeff}}{\text{Factor of Lead Coeff}}$

Possible Rational Zeros: $\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{6}{1}, \pm\frac{6}{2}$

$\{\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm 3/2, \pm 6\}$

EXAMPLE Show that $x=2$ is a zero of $f(x)=2x^3-11x+6$. Then find all the zeros of f.

SOLUTION

$$f(2)=2(2)^3-11(2)+6$$
$$=2(8)-22+6$$
$$=16-22+6$$
$$=-6+6=0$$

$$f(x)=(x-2)\,Q(x)$$

DIVISION

$$\begin{array}{r|rrrr} 2 & 2 & 0 & -11 & +6 \\ & \downarrow & 4 & 8 & -6 \\ \hline & 2 & 4 & -3 & 0 \end{array}$$

R=0 ✓

$$f(x)=(x-2)(2x^2+4x-3)$$
$$=(x-2)\cancel{(2x-3)(x+1)}$$ ~~Not~~ Give up.

$$2x^2+4x-3=0$$
$$x=\frac{-4\pm\sqrt{4^2-4(2)(-3)}}{2(2)}$$
$$=\frac{-4\pm\sqrt{16+24}}{4}$$
$$=\frac{-4\pm\sqrt{40}}{4}=\frac{-4\pm\sqrt{4\cdot10}}{4}$$
$$=\frac{-4\pm2\sqrt{10}}{4}=\frac{\cancel{2}(-2\pm\sqrt{10})}{\cancel{2}\cdot2}$$

$\boxed{x=2}$ $\boxed{x=\frac{-2\pm\sqrt{10}}{2}}$

EXAMPLE: Find all of the real and imaginary zeros of the polynomial.

$$f(x) = x^3 - x^2 - 7x + 15$$

SOLUTION

Step 1 List all possible Rat. Zeros.

factors of 15: $\pm 1, \pm 3, \pm 5, \pm 15$

factors of 1: ± 1

Poss Rat Zeros: $\pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{5}{1}, \pm\frac{15}{1}$

$$\{\pm 1, \pm 3, \pm 5, \pm 15\}$$

Step 2 check to see if any are zeros.

$x = 1$

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -7 & 15 \\ & & 1 & 0 & -7 \\ \hline & 1 & 0 & -7 & 8 \end{array}$$

R=8 no

$x = -1$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -7 & 15 \\ & & -1 & 2 & 5 \\ \hline & 1 & -2 & -5 & 20 \end{array}$$

R=20 NO

$x = 3$

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -7 & 15 \\ & & 3 & 6 & -3 \\ \hline & 1 & 2 & -1 & 12 \end{array}$$

no

$$\boxed{\{-3, 2 \pm i\}}$$

$x = -3$

$$\begin{array}{r|rrrr} -3 & 1 & -1 & -7 & +15 \\ & & -3 & 12 & -15 \\ \hline & 1 & -4 & +5 & 0 \end{array}$$

yes R=0

$f(x) = (x - {-3})(x^2 - 4x + 5)$

$x^2 - 4x + 5 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$

$$= \frac{2(2 \pm i)}{2} = 2 \pm i$$

§3.4 Miscellaneous Equations
HW §3.4 #1-55 odd

<u>EXAMPLE</u> Solve by factoring.

$$x^3+3x^2+x+3=0.$$

<u>SOLUTION</u> Factor by grouping.

$$(x^3+3x^2) + (x+3) = 0$$
$$x^2(x+3) + 1(x+3) = 0$$
$$(x+3)(x^2+1) = 0$$
$$\boxed{x=-3} \qquad x^2+1=0$$
$$x^2=-1$$
$$x=\pm\sqrt{-1}$$
$$x=\pm i$$

$$\{-3, \pm i\}$$

<u>EXAMPLE</u>: $2x^5=16x^2$

<u>SOLUTION</u> $2x^5-16x^2=0$

$$2x^2(x^3-8)=0$$
$$x^2=0, \quad x^3-8=0$$
$$x=0 \qquad x^3-2^3=0$$
$$(x-2)(x^2+2x+2^2)=0$$

Aside
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$x=0,\ x=2, \qquad x^2+2x+4=0$$
$$x=\frac{-2\pm\sqrt{2^2-4\cdot1\cdot4}}{2}$$

$$x = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm \sqrt{4 \cdot 3 \cdot -1}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}\, i}{2}$$

$$= \frac{2(-1 \pm \sqrt{3}\, i)}{2}$$

$$x = -1 \pm \sqrt{3}\, i$$

$$\{0, 2, -1 \pm \sqrt{3}\, i\}$$