Summations

A convenient way of writing sums uses the Greek letter Σ (capital sigma) and is called **sigma notation**.

Let's look at the following series of numbers.

$$1 + 4 + 9 + 16 + 25 + 36$$

We see that the terms of the series follow a pattern. We can rewrite the series as

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

Each term is of the form i^2, where i is a variable that takes values $1, 2, 3, 4, 5$, or 6. A convenient and shorter way to write this series is as

$$\sum_{i=1}^{6} i^2$$

The Greek letter Σ (upper case sigma) corresponds to the letter S in the roman alphabet, and S is the first letter in the word "Sum." Below the sigma is the starting number for the variable i; in this case, i starts at 1. On the top of the sigma, we write the ending value for the number i; in this case, i stops at 5. The variables commonly used in summation notation are i, j, k, l, m, and n. The variables typcally only take integer values.

Thus we write

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$$

Similarly we write

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2$$

or also

$$\sum_{i=1}^{n} \sin(i) = \sin(1) + \sin(2) + \ldots + \sin(n)$$

If f is any function, we would write

$$\sum_{i=1}^{n} f(k) = f(1) + f(2) + f(3) + \ldots + f(n)$$

Example Find the sum.

$$\sum_{i=1}^{4}(2i^2+1)$$

SOLUTION

$$\sum_{i=1}^{4}(2i^2+1) = (2(1)^2+1)+(2(2)^2+1)+(2(3)^2+1)+(2(4)^2+1) = 3+9+19+33 = 64$$

Example Write the sum in expanded form. Do not evaluate the sum.

$$\sum_{k=0}^{6}(2k+1)$$

SOLUTION

$$\sum_{k=0}^{6}(2k+1) = (2\cdot0+1)+(2\cdot1+1)+(2\cdot2+1)+(2\cdot3+1)+(2\cdot4+1)+(2\cdot5+1)+(2\cdot6+1)$$

Example Write the series using summation notation.

$$2+4+6+8+10+12+14$$

SOLUTION

$$\begin{aligned} &2+4+6+8+10+12+14 \\ &\quad= 2(1)+2(2)+2(3)+2(4)+2(5)+2(6)+2(7) \\ &\quad= \sum_{i=1}^{7} 2i \end{aligned}$$

The following are properties of summations. They are essentially the distributive, commutative, and associative properties.

Theorem For sums $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots a_n$, and $\sum_{i=1}^{n} b_i = b_1 + b_2 + b_3 + \ldots b_n$, and c any constant, then

1. $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

2. $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

PROOF Let's prove the first formula.

$$\begin{aligned}\sum_{i=1}^{n} ca_i &= ca_1 + ca_2 + ca_3 + \ldots + ca_n \\ &= c(a_1 + a_2 + a_3 + \ldots + a_n) \\ &= c\sum_{i=1}^{n} a_i\end{aligned}$$

So we see that this is just a restatement of the distributive property.

Suppose we want to find the sum of all of the integers from 1 to 1000. This might take some time.

$$\sum_{i=1}^{1000} i = 1 + 2 + 3 + 4 + \ldots + 999 + 1000$$

We can use a formula to save us time.

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

For the sum $\displaystyle\sum_{i=1}^{1000} i$, we have $n = 1000$, so we get

$$\sum_{i=1}^{1000} i = \frac{1000(1000+1)}{2} = 5,000,500$$

Let's write down a list of formulas that we can use to evaluate sums.

Formulas

1. $\displaystyle\sum_{i=1}^{n} 1 = n$. That is,

   $$\underbrace{1 + 1 + 1 + \ldots + 1}_{n \text{ times}} = n$$

2. $\sum_{i=1}^{n} c = cn$, where c is a constant. That is,

$$\underbrace{c + c + c + \ldots + c}_{n \text{ times}} = nc$$

3. $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

4. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

5. $\sum_{i=1}^{n} i^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

We will prove these formulas using induction in the next section.

Example Evaluate the sum $\sum_{i=1}^{100} i^2$.

Solution We use the formula given above, where $n = 100$.

$$\sum_{i=1}^{100} i^2 = \frac{100(100+1)(2 \cdot 100 + 1)}{6} = 338,350$$

Example Find the value of the sum.

$$\sum_{i=1}^{100} (2i^3 + 5i^2 - 6i + 7)$$

Solution We have $n = 100$.

$$\begin{aligned}
&\sum_{i=1}^{100} (2i^3 + 5i^2 - 6i + 7) \\
&= \sum_{i=1}^{100} 2i^3 + \sum_{i=1}^{100} 5i^2 + \sum_{i=1}^{100} (-6i) + \sum_{i=1}^{100} 7 \\
&= 2\sum_{i=1}^{100} i^3 + 5\sum_{i=1}^{100} i^2 - 6\sum_{i=1}^{100} i + 7\sum_{i=1}^{100} 1
\end{aligned}$$

$$
\begin{aligned}
&= 2\left[\frac{100(100+1)}{2}\right]^2 + 5\left[\frac{100(100+1)(2\cdot 100+1)}{6}\right] - 6\left[\frac{100(100+1)}{2}\right] + 7(100) \\
&= 52,667,150
\end{aligned}
$$

Exercises

Write the sum in expanded form. Do not evaluate the sum.

1. $\sum_{i=1}^{5} \sqrt{i}$

2. $\sum_{i=1}^{6} 3^i$

3. $\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

4. $\sum_{i=1}^{n} i^{10}$

Write the sum in sigma notation.

5. $1+2+3+4+\ldots+10$

6. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots\frac{19}{20}$

7. $2+4+6+\ldots+2n$

8. $1+2+4+8+16+32$

9. $x+x^2+x^3+\ldots+x^n$

Find the value of the sum.

10. $\sum_{i=1}^{50} i$

11. $\sum_{i=1}^{20} 3i^2$

12. $\sum_{i=1}^{40} 5i^2 - 3i + 2$

13. $\sum_{k=1}^{80} 5i^3 + 3i^2 - 5i + 1$

14. $\sum_{i=20}^{100} i$

15. $\sum_{i=30}^{200} i^2$